Stationary Black Holes as Holographs*

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Abstract

In a recent paper (see Ref. [26]) the present author introduced the concept of spacetime conjugation with the help of which the most important gap of the black hole rigidity argument could be filled up. This paper is to provide another use of this concept in general relativity. Here, attention will be restricted to the case of smooth (non-degenerate) four dimensional electrovac stationary black hole spacetimes in Einstein's theory of gravity. By making use of a combination of the Newman-Penrose formalism and that of the null characteristic initial value formulation of Friedrich, we shall determine first the necessary and sufficient condition guaranteeing the existence of globally well-defined Gaussian null coordinate systems in the domain of outer communication. This justifies the claim of [26], that in certain cases the associated domain of outer communication can be smoothly foliated by a 1-parameter family of null hypersurfaces, each generated by congruences of shear free null geodesics. To show the existence of globally "well-behaving" Gaussian null coordinates, as an interesting new result, we also prove that the domain of dependence associated with the selected class of spacetimes—in the characteristic initial value problem based on two null hypersurfaces intersecting in a 2-dimensional spacelike surface—is larger then it has been known to be before, it contains at least a full four dimensional elementary spacetime neighbourhood of the initial data surface. In addition, it is also shown that, within the proposed new framework, the entire of the configuration space of the stationary black hole spacetimes can be investigated in a natural way. It is proved that the geometry of each of these black hole spacetimes is unique, and the only freedom we have—once an appropriate gauge choice has been made—is to specify the geometry of the 2-dimensional space of the null geodesic generators of the event horizon. Accordingly, these stationary black hole spacetimes are nothing but general relativistic holographs. Finally, it is also shown that the necessary and sufficient conditions guaranteeing the existence of global Gaussian null coordinates—in this case the corresponding "holographs" can also be considered to be global—are in a direct correspondence with the criteria selecting the known "unique" asymptotically flat or asymptotically (locally) anti-de-Sitter stationary (electrovac) black hole spacetimes.

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1 Introduction

In our previous paper [26] we introduced the concept of spacetime conjugation with the help of which the black hole rigidity problem has been solved (locally). More concretely, in [26] smooth (non-degenerate) stationary black hole configurations were considered in Einstein's theory of gravity so that the dimension of the spacetimes was allowed to be arbitrary, $n \geq 3$, moreover, quite generic matter fields were included and a non-zero cosmological constant was also admitted. Then, under the assumptions of some mild technical conditions, including e.g. that the space of null generators of the event horizon is compact, the existence of a horizon Killing vector field in a neighbourhood *on both sides* of the event horizon was justified in the smooth setting. We also showed that whenever the applied Gaussian null coordinates can be shown to cover the entire domain of outer communication then the horizon Killing vector field exists globally. Since then we immediately have a generalisation of Hawking's black hole rigidity theorem [13, 14] from the analytic to the smooth setting when we started the investigations described in this paper our primary aim was to identify those conditions which can guarantee the global existence of the applied Gaussian null coordinate systems.

According to our original plans, in this paper, we restrict our attention to four dimensional electrovac black hole spacetimes, however, we shall droop the requirements we had on mind to apply with respect to the asymptotic structure of these spacetimes. In particular, we shall *not* restrict our considerations to asymptotically flat or asymptotically (locally) anti-de-Sitter spacetimes. Our investigations showed—definitely contrary to any of our former anticipations— that the existence of the desired global Gaussian null coordinate systems is, apparently, related to everything in the physics of the associated spacetimes. In other words, once we have answered our original question we can deduce answers to most of the questions in the limited context of the investigated stationary black hole spacetimes. Surprisingly, as a "byproduct" of our analysis a completely new type of black hole uniqueness proof has emerged, although both the notion clarifying a spacetime to be a stationary black hole, as well as, the notion of uniqueness seem to get into a wider context as they have ever been considered to belong to. The applied method, which is a suitable combination of the Newman-Penrose formalism [17] and the formulation of null characteristic initial value problem as it was worked out in details by Friedrich [6], appears to be suitable to explore all the true physical degrees of freedom we have in context of the investigated stationary electrovac black holes. To be able to do so we also needed to invent a new an approach which is significantly different from the standard one, i.e., in its spirit it is definitely counter rational.

The standard approach expresses a somewhat timorous attitude. A bit exaggeratedly we can say that as black holes can be, and therefore are, considered as really harmful and strange objects it has always been assumed implicitly that as far as they are far away from us we are safe. In such a case we might even try to pretend that from gravitational point of view the effect of a black hole can be replaced by that of an "ordinary" star of the same mass. Accordingly, the dominant part of the standard investigations were influenced implicitly by the strategy of approaching a black hole from the distance if possible with having a rest "at infinity" before getting inward. This sort of common sense is expressed the most strikingly in the "topological censorship" arguments where it is, in general, said, e.g., that "Hence the topology of the black holes is controlled by the topology at infinity." [5, 16, 10, 11, 12]. Although we have to admit that we have found one single paper in the literature containing the following statement in the discussion part: "In the Introduction, we offered the view that the topology of the boundary at infinity constrained that of the horizons, but one could equally well reverse the picture." [11]. Nevertheless, such a reverse has not been occurred yet.

In our approach we do reverse the direction of the investigations. Nevertheless, we would like to emphasise that the proposed new setup is suitable to investigate the entire of the black hole spacetime, including thereby the inner black hole region, as well. The basic idea of our approach is very close in spirit to the idea of quasi-local investigations. However, it is not fully quasi-local because, according to our basic geometrical assumption, we start our investigations relying on what we have in an "elementary spacetime region" of a stationary black hole spacetime. In virtue of the results of [26], in the generic case, such an elementary spacetime region contains a bifurcate Killing horizon comprised by two null hypersurfaces each generated by geodesically complete expansion and shear free null congruences. Since the null geodesic generators of the horizon are geodesically complete such an elementary spacetime region has to be global to the associated null directions. We would like to emphasise that requiring the existence of such an elementary spacetime region is not as restrictive as it might seem to be since all the "well-known", asymptotically flat or asymptotically (locally) anti-de-Sitter, static or stationary axisymmetric black hole spacetimes are known to possess such an elementary spacetime neighbourhood. Notice, however, that this generic layout in principle might allow the presence of other type of spacetimes which are either not known or are different in various physical properties from the customary static or stationary axisymmetric black hole spacetimes. By applying the powerful framework of the Newman-Penrose formalism and Friedrich's results—the latter are relevant for the null characteristic initial value formulation of general relativity—, moreover, by starting with what we can rely on possessing an elementary spacetime region, we shall try to get a definite answer about the global existence of the desired Gaussian null coordinate system in the domain of outer communications. We would like to emphasise that we do not presume any sort of asymptotic properties of the associated spacetimes. On the contrary, rather, we would like to use the field equations to read off the asymptotic properties if such a region exists at all. Surprisingly,—or rather it would be more appropriate to say after getting through the associated investigations "Clearly,"—the new approach will provide us some unexpected lessons for free even about those black hole configurations (e.g., about the members of the Kerr family in the vacuum case) which, by any of the confident enough members of the relativity community, are considered to be very well understood.

As an interesting additional new result we show that the domain of dependence associated with the vacuum (or electrovac) Einstein's equations—in the characteristic initial value problem, where an initial data surface consists of a pair of null hypersurfaces intersecting on a 2-dimensional spacelike surface—is much larger then we anticipated before. It is proved that the domain of dependence always contains a full spacetime neighbourhood of the surface of intersection.

This paper is organised as follows: In Section 2 we specify the class of stationary electrovac black hole spacetimes to which our main results apply. Section 3 starts by providing an introduction of the notions of elementary spacetime regions and Gaussian null coordinates. Then some details of the applied mathematical techniques and the relevant results will be recalled, moreover, our new result about the domain of dependence will be presented. This part is followed by an immediate application of the associated techniques to the selected class of stationary black hole spacetimes. Section 4 is to summarise some of the new results relevant for the black hole uniqueness argument, while in Section 5 the consequences of the differences which will show up in the electrovac case will be discussed. Section 6 contains our final remarks and the addressing of some open issues.

2 The stationary electrovac black hole spacetimes

Throughout this paper a spacetime (M, g_{ab}) is taken to be a smooth, paracompact, connected, orientable manifold M endowed with a smooth Lorentzian metric g_{ab} of signature $(+,-,-,-,-)$. It is assumed that (M, g_{ab}) is time orientable and that a time orientation has been chosen.

Consider now a spacetime (M, g_{ab}) with electromagnetic field, the latter is represented by the

2-form F_{ab}, so that the Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2} g_{ab} R + \widetilde{\Lambda} g_{ab} = 8\pi T_{ab}, \tag{2.1}$$

$$\nabla^a F_{ab} = 0 \quad \text{and} \quad \nabla_{[a} F_{bc]} = 0, \tag{2.2}$$

hold, where $\widetilde{\Lambda}$ stands for the cosmological constant, moreover, the energy momentum tensor of the electromagnetic field is given as

$$T_{ab} = -\frac{1}{4\pi} \left[F_{ea} F_b{}^e - \frac{1}{4} g_{ab} \left(F_{ef} F^{ef} \right) \right], \tag{2.3}$$

which automatically satisfies the dominant energy condition.

We shall assume that the spacetime (M, g_{ab}) admits a (global) one-parameter group of isometries, ϕ_t, generated by a Killing vector field t^a, and, as in [26], we assume the existence of a point $p \in M$ so that t^a is (future directed) timelike at p. Then, (M, g_{ab}) possesses a ϕ_t-invariant region of stationarity, $M_{stac} = \phi\{\mathcal{Q}_p\}$, where $\mathcal{Q}_p$ is a sufficiently small open neighbourhood of p so that t^a is timelike in $\mathcal{Q}_p$. The black and white hole regions, $\mathcal{B}$ and $\mathcal{W}$, are defined to be the complement of $I^-[M_{stac}]$ and $I^+[M_{stac}]$, respectively. As in [22, 9, 23, 26] we require that (M, g_{ab}) possess a black hole but no white hole, i.e.

$$M = I^+[M_{stac}]. \tag{2.4}$$

Moreover, the domain of outer communications, $\mathcal{D}$, associated with M_{stac} is defined to be

$$\mathcal{D} = I^-[M_{stac}]. \tag{2.5}$$

The (future) event horizon of the spacetime is defined by

$$\mathcal{N} = \partial I^-[M_{stac}]. \tag{2.6}$$

We assume that $\mathcal{N}$ is smooth, moreover, ϕ_t has no fixed point on $\mathcal{N}$. Furthermore, the space of the null geodesic generators of the event horizon, $\mathcal{N}$, is required to possess the topology of a 2-sphere. Recall that in case of an asymptotically flat four dimensional stationary black hole spacetime there is another result due to also Hawking [13] asserting that, whenever the dominant energy condition is satisfied, $\mathcal{Z}$ necessarily possesses the topology S^2. This result of Hawking has been generalised to the case of asymptotically (locally) anti-de-Sitter spacetimes (see, e.g., [11]).

We shall assume, furthermore, that there are no "holes" in $\mathcal{D}$, i.e., all inextendible future directed null geodesic emanating from a point of $\mathcal{D}$ either cross the event horizon, and enter to and remain in the black hole region, or either of the following two cases happens. The null geodesics in question can be extended to arbitrarily large values of their affine parameter, or they terminate on a curvature singularity. Since our investigations adopt the strategy of trying to go from the inside to the outside as far as the field equations allow to do so the above requirements (which in certain respects are widening even the setup of [26]) seem to be acceptable. Accordingly, "a priory" we do not assume any sort of asymptotic behaviour, not even the existence of an asymptotic region of the black hole spacetimes we study in this paper. It can be seen that the whenever an asymptotic region exists the relevant asymptotic properties do, in fact, follow from the field equations.

Consider now a spacetime (M, g_{ab}) satisfying all of the above assumptions. Then, in virtue of a suitable combination of the results of [9, 23, 26], we have that either t^a itself is normal to the horizon, or in a sufficiently small open neighbourhood, $\mathfrak{O}$, of the event horizon there exists an additional horizon Killing vector field, $\mathfrak{K}^a$, which is normal to $\mathcal{N}$ and with respect to which the electromagnetic field is also invariant. By making use of a suitable orientation it can be ensured that the null vector field $\mathfrak{K}^a$ is future directed and "increasing", along the null geodesic generators of $\mathcal{N}$, to which it is tangent, thereby the quantity called surface gravity[1] $\kappa_\circ$ defined by the relation

$$\mathfrak{K}^a \nabla_a \mathfrak{K}^b = \kappa_\circ \mathfrak{K}^b, \tag{2.7}$$

will be non-negative everywhere on $\mathcal{N}$. By making use of the Einstein's equations, along with the dominant energy condition the value of $\kappa_\circ$ can be shown to be constant throughout $\mathcal{N}$ (see, e.g., [22, 23]). To exclude certain technical difficulties which emerge whenever $\kappa_\circ = 0$, hereafter, we shall assume that $\kappa_\circ$ is a positive constant, in which case both the event horizon and the associated black hole spacetime are called to be non-degenerate.

Appealing then to the results of [21, 22, 9, 23, 26] it can be shown that to any (non-degenerate) stationary electrovac black hole spacetime (M, g_{ab}) it is always possible to find a sufficiently small open neighbourhood $\mathfrak{O}$ of the event horizon, $\mathcal{N}$, so that the subspacetime $(\mathfrak{O}, g_{ab}|_{\mathfrak{O}})$, considered now as a spacetime on its own right, can be extended. In particular, the existence of a smooth extension $(\mathcal{O}^*, g^*_{ab})$ of $(\mathfrak{O}, g_{ab}|_{\mathfrak{O}})$ can be shown so that $(\mathcal{O}^*, g^*_{ab})$ possesses a bifurcate null surface, $\mathcal{H}^*$—i.e., $\mathcal{H}^*$ is the union of two null hypersurfaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, which intersect on a 2-dimensional spacelike surface, $\mathcal{Z}$—such that $\mathcal{N}$ corresponds to the portion of $\mathcal{H}_1$ that lies to the causal future of $\mathcal{Z}$. Furthermore, the vector field $\mathfrak{K}^a$ and the electromagnetic field F_{ab} extend from $\mathfrak{O}$ to fields $\mathfrak{K}^{*a}$ and F^*_{ab} on $\mathcal{O}^*$ so that the Lie derivatives $\mathcal{L}_{\mathfrak{K}^*} g^*_{ab}$ and $\mathcal{L}_{\mathfrak{K}^*} F^*_{ab}$ vanish identically on $\mathcal{O}^*$.

Definition 2.1 *A spacetime which satisfy all of the above assumptions will be referred as (non-degenerate) stationary electrovac black hole spacetime.*

3 Further details of the geometrical setup

Consider now the smooth extension $(\mathcal{O}^*, g^*_{ab})$ of a subspacetime $(\mathfrak{O}, g_{ab}|_{\mathfrak{O}})$ of a (non-degenerate) stationary electrovac black hole spacetime which possesses a bifurcate type event horizon $\mathcal{H}^*$ with bifurcation surface, $\mathcal{Z}$ having the topology of a 2-sphere. The null generators of both of the null hypersurfaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, comprising $\mathcal{H}^*$, are geodesically complete, i.e., they extend arbitrary large values of their affine parameters. Denote by $\mathfrak{U}$ the Killing parameter associated with the horizon Killing vector field $\mathfrak{K}^a$ on $(\mathfrak{O}, g_{ab}|_{\mathfrak{O}})$. An affine parameter u of the null geodesic generators of $\mathcal{N} = J^+[\mathcal{Z}] \cap \overline{\mathcal{D}}$ can then be given as

$$u = e^{\kappa_\circ \mathfrak{U}}. \tag{3.1}$$

The associated parallelly propagated tangent vector field $k^a = (\partial/\partial u)^a$ on $\mathcal{N}$ can be related to $\mathfrak{K}^a$ as

$$k^a = \frac{1}{\kappa_\circ} e^{-\kappa_\circ \mathfrak{U}} \mathfrak{K}^a = \frac{1}{\kappa_\circ u} \mathfrak{K}^a. \tag{3.2}$$

According to this choice the affine parameter u takes the positive values, $u > 0$ on $\mathcal{N}$, while the associated Killing parameter $\mathfrak{U}$ runs from $-\infty$ to ∞. Nevertheless, u can be extended immediately onto $\mathcal{H}_1$ so that it will be an affine parameter everywhere along the null geodesic generators of $\mathcal{H}_1$ and also synchronised so that $u = 0$ corresponds to $\mathcal{Z}$.

3.1 Elementary spacetime neighbourhoods

Consider, now, the 1-parameter family of smooth cross-sections $\mathcal{Z}_u$ of $\mathcal{H}_1$ defined so that $\mathcal{Z}_u$ is comprised by points with $u = const$. Choose then l^a to be the unique *future directed* null vector

[1] Surface gravity is denoted by $\kappa_\circ$ not to be confused with the spin-coefficient κ of the Newman-Penrose formalism [17] which will be applied extensively in our later discussions.

field on $\mathcal{H}_1$ which is everywhere orthogonal to the 2-dimensional cross-sections $\mathcal{Z}_u$ and satisfies the normalising condition $l^a k_a = 1$ everywhere on $\mathcal{H}_1$. Consider now the null geodesics starting at the points of $\mathcal{H}_1$ with tangent l^a. Since $\mathcal{N}$ was assumed to be smooth and, due to the extension procedure, the hypersurface $\mathcal{H}_1$, along with the vector fields k^a and l^a on $\mathcal{H}_1$, are also smooth, these geodesics do not intersect in a sufficiently small open neighbourhood $\mathcal{O} \subset \mathcal{O}^*$ of $\mathcal{H}_1$. Such a neighbourhood $\mathcal{O}$ of $\mathcal{H}_1$ will be referred as "*elementary spacetime region*". By choosing r to be the affine parameter along the null geodesics starting at the points of $\mathcal{H}_1$ with tangent l^a and synchronised so that $r = 0$ on $\mathcal{H}_1$ we get a smooth real function $r : \mathcal{O} \to \mathbb{R}$. The function $u : \mathcal{H}_1 \to \mathbb{R}$, which is smooth by construction, can also be smoothly extended onto $\mathcal{O}$ by requiring its value to be constant along the null geodesics with tangent $l^a = (\partial/\partial r)^a$. Let us denote also by k^a the associated "coordinate basis field", i.e., $k^a = (\partial/\partial u)^a$.

We would like to point to the fact that apparently there is a schizophrenic attitude in defining an elementary spacetime region within $\mathcal{O}^*$ which is itself, by the construction of the spacetime extension, an elementary spacetime region (see for the details of the associated construction in [21, 22]), although, at that time we did not used the terminology "elementary spacetime region". Nevertheless, we believe that the notion of elementary spacetime region—as one of the basic ingredients in the concept of spacetime conjugation—will be applied frequently in various investigations within relativistic physics thereby we decided to repeated the basic elements here.

3.2 Gaussian null coordinates

Now, based on the smooth null hypersurface $\mathcal{H}_1$, and the functions $u, r : \mathcal{O} \to \mathbb{R}$ already defined on $\mathcal{O}$, Gaussian null coordinates (u, r, x^3, x^4) can be defined on suitable subsets $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, which comprise by themselves "elementary spacetime neighbourhoods" of certain subsections, $\widetilde{\mathcal{H}}_1$, of $\mathcal{H}_1$ as follows. Let us choose, first, $\widetilde{\mathcal{Z}}$ to be a connected open subset of the bifurcation surface $\mathcal{Z}$ on which local coordinates (x^3, x^4) can be defined. Choose, furthermore, $\widetilde{\mathcal{H}}_1$ be that subset of $\mathcal{H}_1$ which is span by the null generators of $\mathcal{H}_1$ through the points of $\widetilde{\mathcal{Z}}$. Extend, then, the functions x^3, x^4, first from $\widetilde{\mathcal{Z}}$ onto $\widetilde{\mathcal{H}}_1$, and, second, from $\widetilde{\mathcal{H}}_1$ to $\widetilde{\mathcal{O}}$—consisting of exactly those points of $\mathcal{O}$ which can be achieved along the null geodesics starting on $\widetilde{\mathcal{H}}_1$ with tangent l^a—so that their values are kept to be constant, first along the generators of $\widetilde{\mathcal{H}}_1$, second along the null geodesics with tangent $l^a = (\partial/\partial r)^a$,

Since by construction the vector field $l^a = (\partial/\partial r)^a$ is everywhere tangent to null geodesics we have that $g_{rr} = 0$ throughout $\widetilde{\mathcal{O}}$. Moreover, we also have that the metric functions g_{ru}, g_{r3}, g_{r4} are independent of the r-coordinate, i.e. $g_{ru} = 1, g_{r3} = g_{r4} = 0$ throughout $\widetilde{\mathcal{O}}$. In addition, as a direct consequence of the above construction, g_{uu} and g_{uA} vanish on $\widetilde{\mathcal{H}}_1$. Hence, within $\widetilde{\mathcal{O}}$, there exist smooth functions f and h_A, with $f|_{\widetilde{\mathcal{H}}_1} = (\partial g_{uu}/\partial r)\,|_{r=0}$ and $h_A|_{\widetilde{\mathcal{H}}_1} = (\partial g_{uA}/\partial r)\,|_{r=0}$, so that the spacetime metric in $\widetilde{\mathcal{O}}$ takes the form

$$\mathrm{d}s^2 = r \cdot f \mathrm{d}u^2 + 2\mathrm{d}r\mathrm{d}u + 2r \cdot h_A \mathrm{d}u\mathrm{d}x^A + g_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{3.3}$$

where g_{AB} are smooth functions of u, r, x^3, x^4 in $\widetilde{\mathcal{O}}$ such that g_{AB} is a negative definite 2×2 matrix, and the uppercase Latin indices take the values $3, 4$.

Hereafter, we shall present our arguments only in domains where Gaussian null coordinates can be defined as above. It worth keeping in mind, however, that an elementary spacetime neighbourhood $\mathcal{O}$ can always be covered by sub-regions $\widetilde{\mathcal{O}}$ of the type selected above where Gaussian null coordinates can be defined. By patching these type of coordinate domains the "local" results derived in one of these coordinate domains can always be seen to be valid in an associated elementary spacetime region.

Some comments are in order to avoid later confusions. The Gaussian null coordinates we have just introduced are significantly different from those we applied in [21, 22, 9, 23, 26] in justifying the extendibility of black hole spacetimes or in proving the existence of a horizon Killing vector field. The coordinates used in all of our previous investigations are going to be refereed as "$(\mathfrak{U}, \mathfrak{R}, x^3, x^4)$". Notice that we have already used $\mathfrak{U}$ as the Killing parameter along the generators of $\mathcal{N}$. Reversely, the coordinates we defined above were used previously, e.g., in Section 5 of [21], and they were referred at that time as "(U, R, x^3, x^4)". The reason we decided to introduced the Gaussian null coordinates with the applied notation is that in what follows we shall frequently refer to results of [17] and [6] both of which work using the above introduced coordinate conventions.

We would like to emphasise that although the Gaussian null coordinates $(\mathfrak{U}, \mathfrak{R}, x^3, x^4)$ are also well-defined in $\widetilde{\mathcal{O}}$. Moreover, the "r" and "$\mathfrak{R}$" coordinates, which are affine parameters along the null geodesics with tangent l^a, can be related by making use of $l^a = g^{ab}\nabla_b u$ and $\mathfrak{L}^a = -g^{ab}\nabla_b \mathfrak{U}$ as

$$\mathfrak{R} = -\kappa_\circ u r, \tag{3.4}$$

where the "past directed" null vector field $\mathfrak{L}^a$ is defined as $\mathfrak{L}^a = (\partial/\partial\mathfrak{R})^a$. Moreover, what follows from the results of [26] is, in fact, that $\mathfrak{K}^a = (\partial/\partial\mathfrak{U})^a$ is the horizon Killing vector field in $\mathcal{O}$. Nevertheless, whenever the existence of a global Gaussian null coordinate system of the type (u, r, x^3, x^4) can be guaranteed in $\mathcal{D}$, according to the above observations, the Gaussian null coordinate system $(\mathfrak{U}, \mathfrak{R}, x^3, x^4)$ also exists everywhere in $\mathcal{D}$, thereby, $\mathfrak{K}^a = (\partial/\partial\mathfrak{U})^a$ immediately gives rise to a globally defined Killing vector field on the entire of the outer communication.

3.3 The Newman-Penrose formalism

In deriving the main result of this paper we are going to apply a combination of the Newman-Penrose formalism [17] and the null characteristic initial value formulation of Einstein's theory of gravity, as it was worked out in details by Friedrich [6] (see also [7, 8] for related investigations). Thereby, it seems to be useful to recall the relation between the geometrical setting associated with the above introduced Gaussian null coordinates and the fundamental layout of the Newman-Penrose formalism [17] and that of [6] which is done in this subsection.

The contravariant form of the metric (3.3) in a Gaussian null coordinate system (u, r, x^3, x^4), covering the part $\widetilde{\mathcal{O}}$ of an elementary spacetime region $\mathcal{O}$, can be given as

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{rr} & g^{rB} \\ 0 & g^{Ar} & g^{AB} \end{pmatrix}. \tag{3.5}$$

Choosing now, as it was done in [17], real-valued functions U, X^A and complex-valued functions ω, ξ^A on $\widetilde{\mathcal{O}}$ such that

$$g^{rr} = 2(U - \omega\bar{\omega}), \quad g^{rA} = X^A - (\bar{\omega}\xi^A + \omega\bar{\xi}^A), \quad g^{AB} = -(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B), \tag{3.6}$$

and setting

$$l^\mu = \delta^\mu{}_r, \quad n^\mu = \delta^\mu{}_u + U\delta^\mu{}_r + X^A\delta^\mu{}_A, \quad m^\mu = \omega\delta^\mu{}_r + \xi^A\delta^\mu{}_A, \tag{3.7}$$

we obtain a complex null tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ in $\widetilde{\mathcal{O}}$. We require that U, X^A, and ω vanish on $\widetilde{\mathcal{H}}_1$ which guaranties that n^a is tangent to the generators of $\widetilde{\mathcal{H}}_1$, $n^a = k^a$ there, moreover, m^a and $\overline{m}^a$ are everywhere tangent to the cross-sections $\widetilde{\mathcal{Z}}_u$ of $\widetilde{\mathcal{H}}_1$. In the following we shall consider the derivatives of functions in the direction of the frame vectors defined above and denote the corresponding operators in $\widetilde{\mathcal{O}}$ by

$$\mathrm{D} = \partial/\partial r, \quad \Delta = \partial/\partial u + U\cdot\partial/\partial r + X^A\cdot\partial/\partial x^A, \quad \delta = \omega\cdot\partial/\partial r + \xi^A\cdot\partial/\partial x^A. \tag{3.8}$$

To simplify the Newman-Penrose equation a part of the remaining gauge freedom can be fixed, as it was already done in [17], by assuming that the tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ is parallelly propagated along the null geodesics with tangent $l^a = (\partial/\partial r)^a$ in $\widetilde{\mathcal{O}}$. These assumptions guarantee that for the spin coefficients, corresponding to this specific choice of complex null tetrad, $\kappa = \pi = \varepsilon = 0$, $\rho = \overline{\rho}$, $\tau = \overline{\alpha} + \beta$ hold everywhere in $\widetilde{\mathcal{O}}$. Moreover, since we have chosen n^a so that $n^e \nabla_e n^a = 0$ along the generators of $\widetilde{\mathcal{H}}_1$ the spin coefficient ν, by its definition, is guaranteed to vanish on $\widetilde{\mathcal{H}}_1$. Finally, since u is an affine parameter along the generators of $\widetilde{\mathcal{H}}_1$, e.g., in virtue of (4.14) of [23], we also have that $\gamma + \overline{\gamma} = 0$ thereon. In this case we can also apply a rotation of the form $m^a \rightarrow e^{i\phi} m^a$, where $\phi : \widetilde{\mathcal{H}}_1 \rightarrow \mathbb{R}$ is a suitably chosen real function, so that the spin coefficient γ will, in turn, vanish everywhere on $\widetilde{\mathcal{H}}_1$.

3.4 The null characteristic formulation

We would like to emphasise that the gauge choices we have made so far are exactly the same as those were used in [6], hence, all of the results of Friedrich's formalism can be applied. Here we start by the investigation of the pure vacuum case. Later, in Section 5, it will be shown how the techniques applied below extend to spacetimes with non-zero cosmological constant and with a source free electromagnetic field.

Recall first that the pertinent Newman-Penrose equations[2], (NP.6.10a)-(NP.6.10h), (NP.6.11a)-(NP.6.11r) and (NP.6.12a)-(NP.6.12h), taking them as first order partial differential equations, with respect to Gaussian null coordinates, (u, r, x^3, x^4) in $\widetilde{\mathcal{O}}$, for the vector valued variable

$$\mathbb{V} = (\xi^A, \omega, X^A, U; \rho, \sigma, \tau, \alpha, \beta, \gamma, \lambda, \mu, \nu; \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) \tag{3.9}$$

are overdetermined simply because there are more equations than unknowns. Nevertheless, as it was proved by Friedrich [6], by taking aside some of the Newman-Penrose equations and taking linear combinations some of them the following "reduced set of vacuum field equations"[3]

$$\mathrm{D}\,\xi^A = \rho\,\xi^A + \sigma\,\overline{\xi}^A \tag{FR.1}$$
$$\mathrm{D}\,\omega = \rho\,\omega + \sigma\,\overline{\omega} - \tau \tag{FR.2}$$
$$\mathrm{D}X^A = \tau\,\overline{\xi}^A + \overline{\tau}\,\xi^A \tag{FR.3}$$
$$\mathrm{D}\,U = \tau\,\overline{\omega} + \overline{\tau}\,\omega - (\gamma + \overline{\gamma}) \tag{FR.4}$$
$$\mathrm{D}\,\rho = \rho^2 + \sigma\,\overline{\sigma} \tag{FR.5}$$
$$\mathrm{D}\,\sigma = 2\rho\,\sigma + \Psi_0 \tag{FR.6}$$
$$\mathrm{D}\,\tau = \tau\,\rho + \overline{\tau}\,\sigma + \Psi_1 \tag{FR.7}$$
$$\mathrm{D}\,\alpha = \rho\,\alpha + \beta\,\overline{\sigma} \tag{FR.8}$$
$$\mathrm{D}\,\beta = \alpha\,\sigma + \rho\,\beta + \Psi_1 \tag{FR.9}$$
$$\mathrm{D}\,\gamma = \tau\,\alpha + \overline{\tau}\,\beta + \Psi_2 \tag{FR.10}$$
$$\mathrm{D}\,\lambda = \rho\,\lambda + \overline{\sigma}\,\mu \tag{FR.11}$$
$$\mathrm{D}\,\mu = \rho\,\mu + \sigma\,\lambda + \Psi_2 \tag{FR.12}$$
$$\mathrm{D}\,\nu = \overline{\tau}\,\mu + \tau\,\lambda + \Psi_3 \tag{FR.13}$$
$$\Delta\,\Psi_0 - \delta\,\Psi_1 = (4\,\gamma - \mu)\,\Psi_0 - 2\,(2\,\tau + \beta)\,\Psi_1 + 3\,\sigma\,\Psi_2 \tag{FR.14}$$

[2] To avoid the steady citation of this fundamental work of Newman and Penrose [17] throughout this paper the equations referred as (NP.6.'a combination of a number & a lowercase letter') are always meant to be the original equations listed as (6.'a combination of a number & a lowercase letter') in [17].

[3] To distinguish these equations from others applied in this paper, we shall label the n^{th} reduced equation as "(FR.n)".

$$\Delta\,\Psi_1 + \mathrm{D}\,\Psi_1 - \delta\,\Psi_2 - \overline{\delta}\,\Psi_0 = (\nu - 4\,\alpha)\,\Psi_0 - 2\,(\mu - \gamma - 2\,\rho)\,\Psi_1 - 3\,\tau\,\Psi_2 - 2\,\sigma\,\Psi_3 \tag{FR.15}$$
$$\Delta\,\Psi_2 + \mathrm{D}\,\Psi_2 - \delta\,\Psi_3 - \overline{\delta}\,\Psi_1 = -\lambda\,\Psi_0 - 2\,(\alpha - \nu)\,\Psi_1 + 3\,(\rho - \mu)\,\Psi_2 - 2\,(\tau - \beta)\,\Psi_3 + \sigma\,\Psi_4 \tag{FR.16}$$
$$\Delta\,\Psi_3 + \mathrm{D}\,\Psi_3 - \delta\,\Psi_4 - \overline{\delta}\,\Psi_2 = -2\,\lambda\,\Psi_1 + 3\,\nu\,\Psi_2 + 2\,(\rho - \gamma - 2\,\mu)\,\Psi_3 + (4\,\beta - \tau)\,\Psi_4 \tag{FR.17}$$
$$\mathrm{D}\,\Psi_4 - \overline{\delta}\,\Psi_3 = -3\,\lambda\,\Psi_2 + 2\,\alpha\,\Psi_3 + \rho\,\Psi_4 \tag{FR.18}$$

These equations, beside constituting a determined system for the vector variable, $\mathbb{V}$, are "as good" as the complete set of the Newman-Penrose equations. More precisely, what was proved by Friedrich (see Theorem 1. of [6]) can be rephrased as.

Theorem 3.1 *Denote by $\mathbb{V}_0$ an initial data set, satisfying the "inner" Newman-Penrose equations on the initial data surface comprised by the pair of intersecting null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\mathbb{V}$ is a solution on the domain of dependence $D[\mathcal{H}_1 \cup \mathcal{H}_2]$ to the reduced vacuum field equations, (FR.1)-(FR.18), then $\mathbb{V}$ is also a solution to the full set of the Newman-Penrose equations. Moreover, the metric, the connection and the curvature tensor determined by such a solution $\mathbb{V}$ are so that the connection will be metric and torsion free, as well as, the curvature tensor which can be built from the Weyl spinor components is the curvature tensor associated with this torsion free connection.*

We would like to emphasise that the condition requiring the initial data $\mathbb{V}_0$ to satisfy the "inner" Newman-Penrose equations on $\mathcal{H}_1 \cup \mathcal{H}_2$ is not as restrictive as it seems to be. Indeed, as it was pointed out by Friedrich, see the argument related to Lemma 1. in [6], if we are given a pair of smooth null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ intersecting on a 2-dimensional spacelike surface $\mathcal{Z}$, some of the Newman-Penrose equations will be "interior equations" on $\mathcal{Z}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Therefore, we may start with a "reduced initial data set", $\mathbb{V}_0^{red}$, which consists of the specification of the Weyl spinor components Ψ_4 on $\mathcal{H}_1$ and Ψ_0 on $\mathcal{H}_2$, moreover, it is required to include the specification of the spin-coefficients $\rho, \sigma, \tau, \mu, \lambda$, along with a vector field ξ^A such that $g^{AB} = -(\xi^A \overline{\xi}^B + \overline{\xi}^A \xi^B)$ is a negative definite metric, on $\mathcal{Z}$. It is argued then that the "inner equations on $\mathcal{Z}$" can be solved algebraically for the rest of the variables listed in $\mathbb{V}$. Moreover, once the components of $\mathbb{V}$ are known on $\mathcal{Z}$ the desired initial data $\mathbb{V}_0$ can be determined on $\mathcal{H}_1$ and $\mathcal{H}_2$ by integrating a sequence of ordinary differential equations—these are the corresponding inner equations—along the null geodesic generators a $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Notice that this $\mathbb{V}_0$, by construction, satisfies all the inner equations as it was assumed in Theorem 3.1 above. The way $\mathbb{V}_0$ is determined in practise—in the particular case of a bifurcate Killing horizon, $\mathcal{H}^* = \mathcal{H}_1 \cup \mathcal{H}_2$—will be illustrated in Section 3.7. What will be important for us in our later investigations is the following result of Friedrich [6].

Lemma 3.1 *Assume that $\mathbb{V}$ is a solution to the Newman-Penrose equations. Denote by $\mathbb{V}_0$ the restriction of $\mathbb{V}$ onto $\mathcal{H}_1 \cup \mathcal{H}_2$, moreover, by $\mathbb{V}_0^{red}$ the corresponding reduced initial data as specified above. Then $\mathbb{V}_0^{red}$, along with the Newman-Penrose equations, determines uniquely the initial data set $\mathbb{V}_0$ on $\mathcal{H}_1 \cup \mathcal{H}_2$.*

In addition to the fact that the above reduced vacuum field equations, (FR.1)-(FR.18), comprise a determined system, by an inspection of the particular form they have when written out in the Gaussian null coordinates (u, r, x^3, x^4) in $\widetilde{\mathcal{O}}$, it can also be justified that they possess the form

$$\mathbb{A}^\mu \cdot \partial_\mu \mathbb{V} + \mathbb{B} = 0, \tag{3.9}$$

where the matrices $\mathbb{A}^\mu$ and $\mathbb{B}$ smoothly depend on $\mathbb{V}$, along with its complex conjugate $\overline{\mathbb{V}}$. Moreover, it can also be seen that the matrices $\mathbb{A}^\mu$ are Hermitian, i.e., $\overline{\mathbb{A}}^{\mu T} = \mathbb{A}^\mu$ and the combination $\mathbb{A}^\mu(n_\mu + l_\mu)$ is positive definite. Thereby, the system comprised by (FR.1)-(FR.18) is a quasilinear symmetric hyperbolic system for which the existence and uniqueness of solutions is guaranteed. Summarising these latter observations the following theorem has been justified (see Theorem 2. of [6]).

Theorem 3.2 *In the characteristic initial value problem to any 'reduced initial data set' there always exists a unique solution to the vacuum Einstein's equations.*

3.5 The domain of dependence of $\mathcal{H}_1 \cup \mathcal{H}_2$

According to the conventional understanding the domain of dependence $D[\mathcal{H}_1 \cup \mathcal{H}_2]$ of the two null hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is considered to be a subset of the causal future, $J^+[\mathcal{Z}]$, and causal past, $J^-[\mathcal{Z}]$, of the bifurcation surface $\mathcal{Z}$, in $\mathcal{O}$. Note, however, that by utilising the concept of "spacetime conjugation", which was introduced by the present author in [26], we my also solve the reduced vacuum field equations for the conjugate spacetime metric $\widehat{g}_{ab}$ in $\mathcal{O}$ by making use of the conjugate $\widehat{\mathbb{V}}_0$ of the initial data $\mathbb{V}_0$ specified on $\mathcal{H}_1 \cup \mathcal{H}_2$ originally for g_{ab}. In a lucky situation, by solving the associated initial value problem for the conjugated spacetime, which has its domain of dependence $\widehat{D}[\mathcal{H}_1 \cup \mathcal{H}_2]$ so that it is complement to $D[\mathcal{H}_1 \cup \mathcal{H}_2]$ in $\mathcal{O}$—i.e., $D[\mathcal{H}_1 \cup \mathcal{H}_2] \cup \widehat{D}[\mathcal{H}_1 \cup \mathcal{H}_2] = \mathcal{O}$ and $D[\mathcal{H}_1 \cup \mathcal{H}_2] \cap \widehat{D}[\mathcal{H}_1 \cup \mathcal{H}_2] = \mathcal{H}_1 \cup \mathcal{H}_2$—we might also be able to find solutions satisfying the original field equations everywhere in an elementary spacetime region.

To see that this, in fact, really happens it is very informative to inspect the effect of the action of spacetime conjugation on the Newman-Penrose variables and equations. Before doing that let us recall first that the map of spacetime conjugation is defined in an elementary spacetime neighbourhood $\mathcal{O}$ in the following way [26]. As we have seen $\mathcal{O}$ can be covered by domains in which Gaussian null coordinates can be defined. Denote by $\widetilde{\mathcal{O}}$ one of these domains and assume that (u, r, x^3, x^4) are Gaussian null coordinates on $\widetilde{\mathcal{O}}$. Assuming then that the line element of the spacetime metric, g_{ab}, can be given given by (3.3), the conjugate metric, $\widehat{g}_{ab}$, is defined by requiring that in $\widetilde{\mathcal{O}}$ its line element possesses the form

$$\mathrm{d}\widehat{s}^2 = r \cdot f \mathrm{d}u^2 - 2\mathrm{d}r\mathrm{d}u - 2r \cdot h_A \mathrm{d}u\mathrm{d}x^A + g_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{3.10}$$

where f, h_A and g_{AB} were assumed to be the same smooth functions of the coordinates (u, r, x^3, x^4) as they were in (3.3). This definition provides an immediate one-to-one correspondence between the spacetimes $(\mathcal{O}, g_{ab})$ and $(\mathcal{O}, \widehat{g}_{ab})$, which implies that by performing the spacetime conjugation on $(\mathcal{O}, \widehat{g}_{ab})$ we get back the original spacetime $(\mathcal{O}, g_{ab})$ we started with. More importantly, the causal structure of the conjugate spacetime $(\mathcal{O}, \widehat{g}_{ab})$ differs significantly from that of $(\mathcal{O}, g_{ab})$. The future light cones of $(\mathcal{O}, \widehat{g}_{ab})$ are "horizontal", pointing towards directions which are spacelike with respect to the causal structure of $(\mathcal{O}, g_{ab})$. Moreover, the null cones with respect to g_{ab} and $\widehat{g}_{ab}$, respectively, touch each others smoothly along the null geodesics with tangents l^a and n^a everywhere on $\mathcal{H}_1 \cup \mathcal{H}_2$ while they are keeping touch the null geodesics with tangent l^a everywhere in an elementary spacetime region $\mathcal{O}$. In particular, the relations $D[\mathcal{H}_1 \cup \mathcal{H}_2] \cup \widehat{D}[\mathcal{H}_1 \cup \mathcal{H}_2] = \mathcal{O}$ and $D[\mathcal{H}_1 \cup \mathcal{H}_2] \cap \widehat{D}[\mathcal{H}_1 \cup \mathcal{H}_2] = \mathcal{H}_1 \cup \mathcal{H}_2$ can clearly be seen to hold in $\mathcal{O}$.

Our comments about the properties of the null cones with respect to g^{ab} and $\widehat{g}^{ab}$, respectively, might get to be more clear once the effect of spacetime conjugation is expressed in terms of contravariant form of the metrics g^{ab} and $\widehat{g}^{ab}$. It is straightforward to check that the spacetime conjugation changes only one single component, namely $g^{ur} = 1$ is replaced by $\widehat{g}^{ur} = -1$ while all the other components are intact. Thereby, without loss of generality, we may also assume that the real-valued functions $\widehat{U}$, $\widehat{X}^A$ and the complex-valued functions $\widehat{\omega}$, $\widehat{\xi}^A$ on $\widetilde{\mathcal{O}}$, associated with the conjugate metric $\widehat{g}^{ab}$, coincide with the original functions determining g^{ab} as given by (3.6). Accordingly, the complex null tetrad $\{\widehat{l}^a, \widehat{n}^a, \widehat{m}^a, \overline{\widehat{m}}^a\}$ in $\widetilde{\mathcal{O}}$ can be given as

$$\widehat{l}^\mu = -\delta^\mu{}_r, \quad \widehat{n}^\mu = \delta^\mu{}_u - U\delta^\mu{}_r + X^A\delta^\mu{}_A, \quad \widehat{m}^\mu = -\omega\delta^\mu{}_r + \xi^A\delta^\mu{}_A, \tag{3.11}$$

while the associated differential operators $\widehat{\mathrm{D}}, \widehat{\Delta}$ and $\widehat{\delta}$ can be given as

$$\widehat{\mathrm{D}} = -\partial/\partial r, \quad \widehat{\Delta} = \partial/\partial u - U \cdot \partial/\partial r + X^A \cdot \partial/\partial x^A, \quad \widehat{\delta} = -\omega \cdot \partial/\partial r + \xi^A \cdot \partial/\partial x^A. \tag{3.12}$$

Notice that the appearance of the negative signs associated with the r-components and the r-derivatives above follows from the fact that—according to the set up of the Newman-Penrose formalism—the vector fields $\widehat{l}^a$ and $\widehat{n}^a$ are required to be future directed and scaled, with respect to $\widehat{g}^{ab}$, so that $\widehat{g}_{ab}\widehat{l}^a\widehat{n}^b = 1$ at the bifurcation surface $\mathcal{Z}$. This, in virtue of $\widehat{n}^a = n^a$ on $\mathcal{Z}$, implies that $\widehat{l}^a = -l^a$ there, and since spacetime conjugation keeps intact the null geodesics starting on $\mathcal{H}_1$ with tangent l^a we have that $\widehat{l}^a = -(\partial/\partial r)^a$ everywhere in the elementary spacetime neighbourhood $\mathcal{O}$.

Notice finally, that the slight formal discrepancies showing up in (3.7), (3.8), (3.11) and (3.12) can also be compensated as follows. Once the conjugate spacetime $(\mathcal{O}, \widehat{g}_{ab})$ is given, we may also perform the coordinate transformation $r \rightarrow -r$, while the other coordinates are kept fixed. The changes induced by this transformation on the complex null tetrad $\{\widehat{l}^a, \widehat{n}^a, \widehat{m}^a, \overline{\widehat{m}}^a\}$ and on the associated differential operators $\widehat{\mathrm{D}}, \widehat{\Delta}$ and $\widehat{\delta}$ make immediately transparent the full covariance of the Newman-Penrose formalism, under the action of the product of spacetime conjugation and the indicated slight coordinate transformation.

The key observations of this subsection can now be summarised as.

Theorem 3.3 *Let $\widehat{\mathbb{V}}$ be a solution to the "conjugated reduced field equations" with initial data $\widehat{\mathbb{V}}_0$, yielded by the conjugation of an initial data $\mathbb{V}_0$. Then the spacetime conjugate of $\widehat{\mathbb{V}}$ is a solution to the original reduced field equation corresponding to the initial data $\mathbb{V}_0$.*

As one of the most important corollary of the above result we may assume, without loss of generality, that the domain of dependence of the characteristic initial value problem, associated with the initial data surface $\mathcal{H}_1 \cup \mathcal{H}_2$, coincides with the entire elementary spacetime region $\mathcal{O}$.

3.6 Further geometrical properties of stationary black hole spacetimes

In returning to the basic problem let us recall that our investigations are restricted to the case stationary black hole spacetimes which are definitely not the most generic configurations to which the above recalled results are known to apply. Thereby, it should not be a surprise that even a "reduced initial data set" will be further restricted.

For instance, it follows from the results of [23] (see Remarks 3.1 and 6.1 of that reference) that, in a stationary black hole spacetime with matter satisfying the dominant energy condition, the bifurcate horizon $\widetilde{\mathcal{H}}^*$ is necessarily expansion and shear free. This, in particular, means that the spin coefficients λ and μ vanish on $\widetilde{\mathcal{H}}_1$, while σ and ρ were shown to be identically zero on $\widetilde{\mathcal{H}}_2$. It was also justified that the horizon Killing vector field $\mathfrak{K}^{*a}$ is a repeated principal null vector of the Weyl and Ricci tensors on $\widetilde{\mathcal{H}}^* = \widetilde{\mathcal{H}}_1 \cup \mathcal{H}_2$. More precisely, it was shown in [23] that the Ricci spinor components Φ_{22} and Φ_{21}, as well as, the Weyl spinor components Ψ_3 and Ψ_4 vanish on $\widetilde{\mathcal{H}}_1$. Similarly, Φ_{00} and Φ_{01}, as well as, Ψ_0 and Ψ_1, vanish on $\widetilde{\mathcal{H}}_2$.

The most important missing clue in [23], beside not being aware of the concept of conjugation which we have just introduced recently [26], was the application of the appropriate gauge choice which is done in this paper and it is justified by the following lemma.

Lemma 3.2 *The spin coefficients τ vanishes on $\widetilde{\mathcal{H}}_1$.*

Proof It follows from the definition τ, along with the facts that $l_a = \nabla_a u$ and $l^a n_a = 1$ in $\widetilde{\mathcal{O}}$, that

$$\tau = n^a m^b \nabla_a l_b = n^a m^b \nabla_b l_a = -l^a m^b \nabla_b n_a \tag{3.13}$$

everywhere in $\widetilde{\mathcal{O}}$. Since $\widetilde{\mathcal{N}}$ is a totally geodesic submanifold of the subspacetime $(\mathfrak{O}, g_{ab}|_{\mathfrak{O}})$ in evaluating the term $m^b\nabla_b n_a$ on $\widetilde{\mathcal{N}}$ the way n^a extends from $\widetilde{\mathcal{N}}$ onto $\widetilde{\mathcal{O}}^+ = \{p \in \widetilde{\mathcal{O}} \,|\, u > 0\}$ does not matter. Thereby, in calculating $m^b\nabla_b n_a$ on $\widetilde{\mathcal{N}}$ the substitution $n^a = 1/(\kappa_\circ u)\cdot \mathfrak{K}^a$, see (3.2), can be applied. Taking into account, then, that $\pounds_{\underset{\sim}{m}} u = \pounds_l u = 0$ everywhere in $\widetilde{\mathcal{O}}$, moreover, that $\mathfrak{K}^a$ is a Killing vector field, i.e. $\nabla_b \mathfrak{K}_a = -\nabla_a \mathfrak{K}_b$ in $\widetilde{\mathcal{O}}^+$, it follows from (3.13) that

$$\tau = -\frac{1}{\kappa_\circ u} l^a m^b \nabla_b \mathfrak{K}_a = \frac{1}{\kappa_\circ u} l^a m^b \nabla_a \mathfrak{K}_b = l^a m^b \nabla_a n_b \tag{3.14}$$

on $\widetilde{\mathcal{N}}$. This relation, along with the fact that n^a is parallel with respect to l^a, justifies then that τ has to vanish on $\widetilde{\mathcal{N}}$. Due to the 'wedge reflection' symmetry of the extension $(\mathcal{O}^*, g^*_{ab})$ τ also vanishes on $\widetilde{\mathcal{H}}_1 \setminus \widetilde{\widetilde{\mathcal{Z}}}$ and, in turn, by continuity, on the entire of $\widetilde{\mathcal{H}}_1$. □

3.7 The determination of a full initial data set

As it was argued above, from a reduced initial data set the full information associated with a solution of the vacuum Einstein's equations can be recovered in the domain of dependence of the initial data surface, i.e., in the entire of the associated elementary spacetime region. Moreover, a reduced initial data set, $\mathbb{V}_0^{red}$, is given as

$$\mathbb{V}_0^{red} = \{\rho, \sigma, \mu, \lambda, \tau\,;\, \xi^A\}|_{\widetilde{\widetilde{\mathcal{Z}}}} \cup \{\Psi_4\}|_{\widetilde{\mathcal{H}}_1} \cup \{\Psi_0\}|_{\widetilde{\mathcal{H}}_2} \tag{3.15}$$

Accordingly, in case of a considered stationary black hole spacetime, we need to specify the spin-coefficients $\rho, \sigma, \mu, \lambda, \tau$ and the vector field ξ^A on $\widetilde{\widetilde{\mathcal{Z}}}$, moreover, Weyl spinor components Ψ_4 on $\widetilde{\mathcal{H}}_1$ and Ψ_0 on $\widetilde{\mathcal{H}}_2$.

As it was pointed out above, in this particular case, the reduced initial data set simplifies considerably since the initial data surface is comprised by two expansion and shear free null geodesic congruences. This, along with lemma 3.2, implies that the only non-trivial quantity which can "yet" be freely specified as our initial data on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$ is nothing but the vector field ξ^A on $\widetilde{\widetilde{\mathcal{Z}}}$.

To illustrate the way a full initial data set is produced from a reduced one, in our simple setting, moreover, to appreciate the robustness of the setup of [6] we shall carry out the determination of a full initial data set $\mathbb{V}_0$, on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$, from the significantly reduced one

$$\mathbb{V}_0^{red} = \{\rho = \sigma = \mu = \lambda = 0\,;\, \xi^A\}|_{\widetilde{\widetilde{\mathcal{Z}}}} \cup \{\Psi_4 = \tau = \nu = \gamma = 0\}|_{\widetilde{\mathcal{H}}_1} \cup \{\Psi_0 = 0\}|_{\widetilde{\mathcal{H}}_2}, \tag{3.16}$$

which is compatible with all of our former observations concerning the geometrical properties of a bifurcate Killing horizon of a stationary black hole, as well as, with all of the gauge conditions we have made.

Let us start by considering the "inner equations" we have on $\widetilde{\widetilde{\mathcal{Z}}}$. Notice first that (NP.6.11k) and (NP.6.11m) immediately implies that both Ψ_1 and Ψ_3 vanishes on $\widetilde{\widetilde{\mathcal{Z}}}$. Furthermore, (NP.6.10f), along with our gauge conditions $\tau = \overline{\alpha} + \beta$ in $\widetilde{\mathcal{O}}$ and, in particular, $\tau = 0$ on $\widetilde{\mathcal{H}}_1$, gives that

$$\delta\overline{\xi}^A - \overline{\delta}\xi^A = 2\,(\overline{\beta}\,\xi^A - \beta\,\overline{\xi}^A). \tag{3.17}$$

This equation can be solved algebraically for β and $\overline{\beta} = -\alpha$ on $\widetilde{\widetilde{\mathcal{Z}}}$. By applying then (NP.6.11l) we immediately get

$$-\,\delta\overline{\beta} - \overline{\delta}\beta = 4\,\beta\,\overline{\beta} - \Psi_2. \tag{3.18}$$

which fixes the value of Ψ_2 on $\widetilde{\widetilde{\mathcal{Z}}}$. Notice that the last relation also imply that Ψ_2 is necessarily real on $\widetilde{\widetilde{\mathcal{Z}}}$.

Consider now the inner equations on $\widetilde{\mathcal{H}}_2$. First of all, since $\Psi_0 \equiv 0$ there (NP.6.12a), along with the fact that $\Psi_1|_{\widetilde{\mathcal{Z}}} \equiv 0$, implies that $\Psi_1 \equiv 0$ on $\widetilde{\mathcal{H}}_2$. Similarly, since $\rho|_{\widetilde{\mathcal{Z}}} \equiv 0$ and $\sigma|_{\widetilde{\mathcal{Z}}} \equiv 0$, (NP.6.11a) and (NP.6.11b) imply that $\rho \equiv 0$ and $\sigma \equiv 0$ on $\widetilde{\mathcal{H}}_2$. The vanishing of ρ, σ and Ψ_1 on $\widetilde{\mathcal{H}}_2$ can then be used, along with (NP.611c-d-e), to conclude that

$$\mathrm{D}\alpha = \mathrm{D}\beta = \mathrm{D}\tau = 0 \tag{3.19}$$

on $\widetilde{\mathcal{H}}_2$, which along with the vanishing of τ on $\widetilde{\mathcal{Z}}$ does imply that $\tau \equiv 0$ on $\widetilde{\mathcal{H}}_2$. Similarly, (NP.6.12b) gives then

$$\mathrm{D}\Psi_2 = 0 \tag{3.20}$$

on $\widetilde{\mathcal{H}}_2$. In virtue of (NP.6.11g) and (NP.6.11i) we also have then that $\lambda \equiv 0$ on $\widetilde{\mathcal{H}}_2$ since λ vanishes on $\widetilde{\mathcal{Z}}$. Two other spin coefficients, γ and μ, can be determined with the help of (NP.6.11f) and (NP.6.11h) which, along with (3.20) and their vanishing on $\widetilde{\mathcal{Z}}$, give that $\gamma = r \cdot \Psi_2$ and $\mu = r \cdot \Psi_2$ on $\widetilde{\mathcal{H}}_2$.

By completely analogous reasoning the inner equations on $\widetilde{\mathcal{H}}_1$, along with the vanishing of ν, γ and τ there, which follow from our gauge choice, can be used to justify the followings. First, since $\Psi_4 \equiv 0$ there (NP.6.12h), along with the fact that $\Psi_3|_{\widetilde{\mathcal{Z}}} \equiv 0$, implies that $\Psi_3 \equiv 0$ on $\widetilde{\mathcal{H}}_1$. Similarly, since $\mu|_{\widetilde{\mathcal{Z}}} \equiv 0$ and $\lambda|_{\widetilde{\mathcal{Z}}} \equiv 0$, (NP.6.11n) and (NP.6.11j) imply that $\mu \equiv 0$ and $\lambda \equiv 0$ on $\widetilde{\mathcal{H}}_1$. The vanishing of μ, λ and Ψ_3 on $\widetilde{\mathcal{H}}_1$ can be used, along with (NP.611r-o-p), to conclude that

$$\Delta\alpha = \Delta\beta = \Delta\sigma = 0 \tag{3.21}$$

on $\widetilde{\mathcal{H}}_1$. This latter relation, along with the vanishing of σ on $\widetilde{\mathcal{Z}}$, implies that $\sigma \equiv 0$ on $\widetilde{\mathcal{H}}_1$. Similarly, (NP.6.12g) gives then

$$\Delta\Psi_2 = 0 \tag{3.22}$$

on $\widetilde{\mathcal{H}}_1$. The only remaining non-trivial spin coefficient is ρ which is determined with the help of (NP.6.11q), along with (3.22) and its vanishing on $\widetilde{\mathcal{Z}}$, as $\rho = -u \cdot \Psi_2$ on $\widetilde{\mathcal{H}}_1$.

To have a full initial data set $\mathbb{V}_0$ on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$, in addition to what we have already derived above, we also need to determine the behaving all of the Weyl spinor components on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$, as well as, the value of ν on $\widetilde{\mathcal{H}}_2$. For instance, from (NP.6.12f) we get that $\Delta\Psi_1 - \delta\Psi_2 = 0$ on $\widetilde{\mathcal{H}}_1$, which, in virtue of $\Psi_1|_{\widetilde{\mathcal{Z}}} = 0$ and the u-independentness of Ψ_2 implies that

$$\Psi_1 = u \cdot \delta\Psi_2 \tag{3.23}$$

on $\widetilde{\mathcal{H}}_1$. By an analogous argument, we also get, from (NP.6.12e) and from what we have just established, that

$$\Psi_0 = \frac{1}{2}u^2 \left(\delta^2\Psi_2 - 2\beta \cdot \delta\Psi_2\right) \tag{3.24}$$

holds on $\widetilde{\mathcal{H}}_1$.

Completely parallel to the reasoning applied in the previous paragraph we can also show, by making use of (NP.6.12c) and (NP.6.12d), that

$$\Psi_3 = r \cdot \overline{\delta}\Psi_2 \tag{3.25}$$

and

$$\Psi_4 = \frac{1}{2}r^2 \left(\overline{\delta}^2\Psi_2 + 2\alpha \cdot \overline{\delta}\Psi_2\right) \tag{3.26}$$

hold on $\widetilde{\mathcal{H}}_2$. Finally, by making use of (NP.6.11i) and (3.25) the value of ν can be determined on $\widetilde{\mathcal{H}}_2$ as

$$\nu = \frac{1}{2} r^2 \cdot \overline{\delta}\Psi_2. \tag{3.27}$$

It is also informative to collect what we have already established on our initial data surface $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$ (see Table 1).

$\widetilde{\mathcal{H}}_1$	$\widetilde{\mathcal{Z}}$	$\widetilde{\mathcal{H}}_2$
$\rho = -u \cdot \Psi_2$	$\rho = 0$	$\rho = 0$
$\mu = 0$	$\mu = 0$	$\mu = r \cdot \Psi_2$
$\sigma = \lambda = \tau = 0$	$\sigma = \lambda = \tau = 0$	$\sigma = \lambda = \tau = 0$
$\Delta\alpha = \Delta\beta = 0$	$\alpha, \beta :\ \tau = \overline{\alpha} + \beta = 0$	$\mathrm{D}\alpha = \mathrm{D}\beta = 0$
$\Delta\Psi_2 = 0$	ξ^A & $\alpha, \beta \rightarrow \Psi_2$	$\mathrm{D}\Psi_2 = 0$
$\Psi_0 = \frac{1}{2} u^2 \left(\delta^2\Psi_2 - 2\beta \cdot \delta\Psi_2\right)$	$\Psi_0 = 0$	$\Psi_0 = 0$
$\Psi_1 = u \cdot \delta\Psi_2$	$\Psi_1 = 0$	$\Psi_1 = 0$
$\Psi_3 = 0$	$\Psi_3 = 0$	$\Psi_3 = r \cdot \overline{\delta}\Psi_2$
$\Psi_4 = 0$	$\Psi_4 = 0$	$\Psi_4 = \frac{1}{2} r^2 \left(\overline{\delta}^2\Psi_2 + 2\alpha \cdot \overline{\delta}\Psi_2\right)$
(gauge) $\nu = \gamma = 0 \quad \rightarrow$	$\nu = \gamma = 0 \quad \rightarrow$	$\nu = \frac{1}{2} r^2 \cdot \overline{\delta}\Psi_2$, $\gamma = r \cdot \Psi_2$

Table 1: The full initial data set $\mathbb{V}_0$, on the intersecting null hypersurfaces $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$.

4 The global existence of shear free congruences

As it was already discussed in [26] the black hole rigidity argument can be closed if our Gaussian null coordinates can be shown to be well-defined everywhere on the entire of the domain of outer communications $\mathcal{D}$.

Since our Gaussian null coordinate system is based on the use of null geodesic congruences it is straightforward to see that these coordinates are globally well-defined in $\mathcal{D}$ unless caustics, associated with conjugate points, can develop along them. To be more specific, consider the 1-parameter family of 2-dimensional spacelike surfaces $\mathcal{Z}_u$ foliating $\mathcal{H}_1$, they are the '$u = const$' slices of $\mathcal{H}_1$ according to the notation introduced in Subsection 3.2. If the null geodesic congruences intersecting $\mathcal{H}_1$ at the 2-dimensional spacelike surfaces $\mathcal{Z}_u$ transversely are non-contracting, at the surfaces $\mathcal{Z}_u$ in the direction of $\mathcal{D}$, moreover, the shear of these geodesics is guaranteed to vanish then by standard results (see, e.g., [14, 19, 20]) we know that these null geodesic congruences will be free of conjugate points. Consequently, the associated Gaussian null coordinates are globally well-defined, and in turn, in case of the considered four dimensional electrovac spacetimes, our argument along with the results of [26], provide an immediate generalisation of the black hole rigidity theorem of Hawking to the corresponding non-analytic setting.

We will see that somewhat more is true but let us concentrate now on the existence of the globally shear free null geodesic congruences in our setting.

Lemma 4.1 *Let $\widetilde{\mathcal{Z}}_u$ be a 1-parameter family of smooth cross-sections of $\widetilde{\mathcal{H}}_1$ with $\widetilde{\mathcal{Z}}_u$ being a smooth 2-dimensional spacelike surface for each constant value of u. Then the null geodesic congruences with tangent l^a meeting $\widetilde{\mathcal{H}}_1$ at the spacelike surfaces $\widetilde{\mathcal{Z}}_u$ orthogonally are non-contracting, in the direction of $\mathcal{D}$, at $\widetilde{\mathcal{H}}_1$ if and only if $\rho \geq 0$ thereon.*

Proof Start by recalling that the spin-coefficient ρ is defined as $\rho = m^a \overline{m}^b \nabla_a l_b$ which, along with $g^{ab} = l^a k^b + k^a l^a - m^a \overline{m}^b - \overline{m}^a m^b$ and that all the tetrad vectors are parallelly propagated with respect to l^a, as well as, l^a is null everywhere, gives that

$$\rho = m^a \overline{m}^b \nabla_a l_b = -g^{ab} \nabla_a l_b = -\theta_{(l)}, \tag{4.1}$$

where $\theta_{(l)}$ denotes the expansion of the null congruence with respect to l^a. Thus, whenever $\rho \geq 0$ on $\widetilde{\mathcal{H}}_1$ the expansion of the outward directed null geodesics geodesics, with tangent vector $-l^a$, is non-negative. $\square$

In virtue of this result in the vacuum case, i.e., whenever $\rho = -u \cdot \Psi_2$, the congruences in question are non-contracting whenever Ψ_2 is guaranteed to be non-positive. Notice also that for all the customary solutions with non-negative mass Ψ_2 always satisfy this requirement.

The key result of this Subsection is formulated by the following lemma.

Lemma 4.2 *Consider the null geodesics with tangent $l^a = (\partial/\partial r)^a$ in a subset $\widetilde{\mathcal{O}}$ of an elementary spacetime region $\mathcal{O}$ where Gaussian null coordinates (u, r, x^3, x^4) are defined. These geodesics are shear-free if and only if*

$$\delta^2 \Psi_2 - 2\beta \cdot \delta \Psi_2 = 0 \tag{4.2}$$

at the bifurcation surface $\mathcal{Z}$.

Proof We shall prove this statement by making use of the null characteristic initial value problem, augmented by the new domain of dependence result of Subsection 3.5, and along with an application of a trick similar in spirit to the one applied in proof of the Goldberg-Sacs theorem by Newman and Penrose [17]. Let us start by assuming that we have a generic system as it was describe in Subsection 3.3. Apply now a null rotation

$$l'^a = l^a, \quad m'^a = m^a + \overline{c} \cdot l^a, \quad n'^a = n^a + c \cdot m^a + \overline{c} \cdot \overline{m}^a + c\overline{c} \cdot l^a \tag{4.3}$$

of the null tetrad $\{l^a, n^a, m^a\}$, where $c : \widetilde{\mathcal{O}} \to \mathbb{C}$ is an arbitrary complex valued function on $\widetilde{\mathcal{O}}$. It is straightforward to check that the spin-coefficient σ, and the Weyl spinor component Ψ_0 are intact, i.e., $\sigma' = \sigma$ and $\Psi'_0 = \Psi_0$, while Ψ_1 transforms as $\Psi'_1 = \Psi_1 + c \cdot \Psi_0$. Moreover, two of the pertinent Newman-Penrose equations, which also belong to the reduced system of Friedrich, take the form

$$\mathrm{D}'\,\sigma' = 2\rho'\,\sigma' + \Psi'_0 \tag{4.4}$$

$$\Delta'\,\Psi'_0 - \delta'\,\Psi'_1 = (4\,\gamma' - \mu')\,\Psi'_0 - 2\,(2\,\tau' + \beta')\,\Psi'_1 + 3\,\sigma'\,\Psi'_2 \tag{4.5}$$

Choose now $c : \widetilde{\mathcal{O}} \to \mathbb{C}$ to be the function which annihilates Ψ'_1 throughout $\widetilde{\mathcal{O}}$. Then, the above two equations, taking them as a first order partial differential equations with respect to Gaussian null coordinates (u, r, x^3, x^4) in $\widetilde{\mathcal{O}}$, comprise a symmetric hyperbolic system for σ' and Ψ'_0 so that the equations are also homogeneous and linear in σ' and Ψ'_0. This, in particular, implies that whenever we have identically zero initial data for them the only unique solution they might possess is the identically zero solution. Thereby, the null geodesics with tangent l^a in $\widetilde{\mathcal{O}}$ are shear-free if and only if the initial data for σ' and Ψ'_0 are trivial. Recall that, in virtue of Table 1 the spin-coefficient σ is identically zero on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$, while Ψ_0 vanishes on $\widetilde{\mathcal{H}}_2$ and it also vanishes on $\widetilde{\mathcal{H}}_1$ whenever

$\delta^2\Psi_2 - 2\beta\cdot\delta\Psi_2 = 0$ along the generators of $\widetilde{\mathcal{H}}_1$, which, in virtue of the u-independentness of β and Ψ_2, holds whenever $\delta^2\Psi_2 - 2\beta\cdot\delta\Psi_2 = 0$ holds on $\widetilde{\mathcal{Z}}$. Appealing now to the invariance of σ and Ψ_0 under the action of a null rotation we have that the initial data for σ' and Ψ_0' is identically zero on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$ whenever $\delta^2\Psi_2 - 2\beta\cdot\delta\Psi_2 = 0$ holds on $\mathcal{Z}$ which completes our argument. □

Before proceeding in clearing up the geometrical meaning of the above necessary and sufficient condition let us recall that whenever (4.2) are satisfied on $\mathcal{Z}$, in consequence of the fact that Ψ_2 is real, we also have that

$$\overline{\delta}^2\Psi_2 - 2\beta\cdot\overline{\delta}\Psi_2 = 0 \tag{4.6}$$

on $\mathcal{Z}$. By a completely parallel argument to the above one it can also be shown then that this relation guaranties the vanishing of the spin-coefficient λ, and the Weyl spinor component Ψ_4 throughout $\widetilde{\mathcal{O}}$. In other words, when we have the desired global 1-parameter family of shear-free null geodesic congruences we immediately have also another one. This situation can be rephrased by saying that whenever (4.2) is satisfied on $\mathcal{Z}$ the corresponding stationary black hole spacetime must be of *Petrov-type D.*

4.1 The geometrical meaning of the necessary and sufficient condition

As we have seen the necessary and sufficient conditions for the existence of the desired shear free null geodesic congruences is the vanishing of $\delta^2\Psi_2 - 2\beta\cdot\delta\Psi_2$ on $\mathcal{Z}$. Therefore it is of obvious interest to know what are the true geometrical implications of this condition.

To have a hint on the possible answers recall first that in the Newman-Penrose formalism the Gaussian curvature, $\mathcal{K}_{\mathcal{G}}$, of the 2-dimensional spacelike surface $\mathcal{Z}$ can be expressed as

$$\mathcal{K}_{\mathcal{G}} = 2\,\mathfrak{Re}\left\{\rho\mu - \sigma\lambda - \Psi_2 + \Phi_{11} + \Lambda\right\}\big|_{\mathcal{Z}}\,. \tag{4.7}$$

This relation in the particular case, with $\rho = \sigma = \mu = \lambda = \Phi_{11} = \Lambda = 0$, we are dealing with implies that $\mathcal{K}_{\mathcal{G}} = -2\Psi_2$. In other words, condition (4.2) simply gives rise to the following simple restriction on the geometry of 2-dimensional spacelike surface $\mathcal{Z}$.

Lemma 4.3 *Ψ_2 satisfies (4.2) if and only if the Gaussian curvature $\mathcal{K}_{\mathcal{G}}$ of $\mathcal{Z}$, with respect to a spherical coordinate system, (θ,ϕ), on $\mathcal{Z}$, is of the form*

$$\mathcal{K}_{\mathcal{G}} = \eta + \chi\cos\theta, \tag{4.8}$$

where η and χ are (real) constants.

Proof Notice first that the $\{p,q\}$-type of $\delta\Psi_2$ is $\{1,-1\}$, which is in accordance with the fact that $\delta\Psi_2$ possesses "*spin-weight*" $s = \frac{1}{2}(p-q) = 1$ and "*boost-weight*" $b = \frac{1}{2}(p+q) = 0$. This implies then, in virtue of (4.12.18) of [19] that

$$\eth\,(\delta\Psi_2) = \delta^2\Psi_2 - \beta\cdot\delta\Psi_2 - \overline{\alpha}\cdot\delta\Psi_2, \tag{4.9}$$

which, along with the fact that $\tau = \overline{\alpha} + \beta = 0$ on $\mathcal{Z}$, justifies that (4.2) is equivalent to

$$\eth\,(\delta\Psi_2) \equiv 0 \tag{4.10}$$

on $\mathcal{Z}$, where $\eth$ stands for the "*edth*"-operator of Newman and Penrose [18]. Appealing now to Prop. 4.15.18 of [19] we know that $\delta\Psi_2$ satisfies (4.10) if and only if $\delta\Psi_2$ is a *spin-weighted* spherical harmonics with $s = l = 1$. In other words, assuming now that spherical coordinates (θ,ϕ) have been chosen on $\mathcal{Z}$, there exist constants $a_m \in \mathbb{C}$, with $m = -1, 0, 1$, so that $\delta\Psi_2$ can be given as the constant linear combination $\delta\Psi_2 = \sum_m a_m \cdot {}_1Y_{1,m}$.

Noticing that Ψ_2 itself is a $\{0,0\}$-type scalar $\delta\Psi_2$ can also be written as $\eth\Psi_2$ which, along with ${}_1Y_{1m} = \frac{1}{\sqrt{2}}\eth Y_{1m}$, which follows from the generic relation between the *spin-weighted* and *ordinary* spherical harmonics, leads to the relation

$$\eth\left(\Psi_2 - \sum_m a_m \cdot Y_{1,m}\right) = 0 \tag{4.11}$$

on $\mathcal{Z}$. By appealing again to Prop. 4.15.18 of [19] this latter relation can be satisfied if and only if the function $\Psi_2 - \sum_m a_m \cdot Y_{1,m}$ is a *spin-weighted* spherical harmonics with $s = l = 0$. As a consequence of all the above partial results, we have that Ψ_2 satisfies (4.2) if and only if it possesses the form

$$\Psi_2 = a_1 \cdot Y_{1,1} + a_0 \cdot Y_{1,0} + a_{-1} \cdot Y_{1,-1} + b \cdot Y_{0,0}. \tag{4.12}$$

Since Ψ_2 have been shown to be real, moreover, $Y_{1,0} = \sqrt{3}/(2\sqrt{\pi})\cos\theta$ and $Y_{0,0} = 1/(2\sqrt{\pi})$ the coefficients a_0 and b have to be real. Furthermore, since $\overline{Y_{1,-1}} = -Y_{1,1} = \sqrt{3}/(2\sqrt{2\pi})\sin\theta\, e^{i\phi}$ the coefficients a_1 and a_{-1} have to be related as $a_{-1} = -\overline{a_1}$ and, in turn, we have that

$$a_1 \cdot Y_{1,1} + a_{-1} \cdot Y_{1,-1} = a_1 \cdot Y_{1,1} + \overline{a_1 \cdot Y_{1,1}} = 2\,\mathfrak{Re}\{a_1\}\,\sin\theta \tag{4.13}$$

is also satisfied. These latter observations imply then that by making use of a coordinate transformation of the form $\theta \to \theta + \theta_0$, with some $\theta_0 \in \mathbb{R}$, as well as, suitably redefining the constants we get that

$$\mathcal{K}_{\mathcal{G}} = -2\Psi_2 = \eta + \chi\cos\theta, \tag{4.14}$$

as we desired to show. □

We would like to emphasise that the constants η and χ are not completely arbitrary either since, e.g., the Gaussian curvature $\mathcal{K}_{\mathcal{G}}$ is subject to the Gauss-Bonnet theorem requiring that

$$\oint_{\mathcal{Z}} \mathcal{K}_{\mathcal{G}} \cdot \epsilon = 4\pi \cdot (1 - g), \tag{4.15}$$

with $g = 0$ for the present case when $\mathcal{Z}$ possesses the topology of a 2-sphere, where ϵ denotes the volume element determined by the 2-metric on $\mathcal{Z}$. This metric can be given in terms of the holomorphic "spherical coordinate"

$$\xi = \log\tan\frac{\theta}{2} + i\phi \tag{4.16}$$

as

$$ds^2_{\mathcal{Z}} = -\frac{2d\xi d\overline{\xi}}{P\overline{P}}, \tag{4.17}$$

where the product $P\overline{P}$ can also be shown to be related (see 4.15.35 of [19]) to the Gaussian curvature as

$$\mathcal{K}_{\mathcal{G}} = P\overline{P}\frac{\partial^2}{\partial\xi\partial\overline{\xi}}\log\left(P\overline{P}\right). \tag{4.18}$$

To find the stationary black hole solutions possessing global Gaussian null coordinate systems we need to solve (4.18) so that the Gaussian curvature $\mathcal{K}_{\mathcal{G}}$ is given by (4.8). Once we know P we may choose ξ_A as

$$\xi_A = -\frac{1}{\overline{P}}\left(d\overline{\xi}\right)_A. \tag{4.19}$$

From this point the associated vacuum solution is fixed and switching from the Gaussian null coordinates (u, r, x^3, x^4) to $(\mathfrak{U}, \mathfrak{R}, x^3, x^4)$ we immediately get the metric in "Finkelstein-type" ingoing null coordinates with $\mathfrak{U}$ as an adapted Killing coordinate, while the original (u, r, x^3, x^4) coordinates are nothing but the associated "Kruskal-type" coordinates. The result of the explicit calculation—including some interesting cases with matter fields—will be published elsewhere.

Let us finally emphasise that the explicit form of the Gaussian curvature $\mathcal{K}_{\mathcal{G}}$ as it is given by (4.8) is very suggestive by itself. It immediately implies that geometry on $\mathcal{Z}$ possesses the axial Killing field, $\phi^a = (\partial/\partial\phi)^a$. Notice, however, that the above form of $\mathcal{K}_{\mathcal{G}}$ is much more restricted than simply being compatible with the existence of an axial Killing vector field on $\mathcal{Z}$. It is also informative to look at the mass and angular momentum aspects of these configurations. It can be seen that whenever

$$\delta\Psi_2 = 0 \tag{4.20}$$

on $\mathcal{Z}$—notice that this relation implies that (4.2) also holds—the constant χ is zero, i.e., the bifurcation surface $\mathcal{Z}$ is a metric sphere and the associated Schwarzschild black hole spacetime does possess only mass but no angular momentum. On the other hand, whenever (4.2) holds but $\delta\Psi_2$ does not vanish we have that the Weyl spinor component Ψ_1 is not identically zero and, e.g., the associated Bondi angular momentum aspect of the vacuum solution is that of the Kerr solution (see, e.g., [28] for a discussion about the Bondi mass and angular momentum aspects).

4.2 The physical implications of the necessary and sufficient condition

Having the geometrical meaning of (4.2)—i.e., the necessary and sufficient condition ensuring the the existence of global foliation of a stationary black hole spacetime by shear free null geodesic congruences—to be understood it seems also to be important to make clear what are the associated true physical implications. In other words, it is important to know what happens in the generic situation whenever (4.2) does not hold.

To have some hints about the possible answers it is rewarding to inspect again for a short while Table 1. What might not be too striking for the first glance is the u-dependence of the Weyl spinor components Ψ_0 and Ψ_1 along the null geodesic generators of $\widetilde{\mathcal{H}}_1$, and similarly, the r-dependence of Ψ_3 and Ψ_4 along the null geodesic generators of $\widetilde{\mathcal{H}}_2$. Of course any of these quantities vanishes at the bifurcation surface but when we approach the asymptotic ends they blow up.

Such a blow up does not occur if $\delta\Psi_2 = \eth\Psi_2 = 0$ on $\mathcal{Z}$ but this condition seems to be very restrictive—according to the above discussion—it is equivalent to requiring $\mathcal{Z}$ to be a metric sphere, and in the vacuum case it is satisfied only by the Schwarzschild solution. Having this point to be made clear the following question immediately emerge. Is it true that even the Kerr solution possesses a property which has not been recognised or emphasised yet? The answer is confirmatory. Notice, however, that the Kerr solution satisfies (4.2) thereby it admits only "mild" type of irregularities since then $\Psi_1 = u \cdot \eth\Psi_2$ along the null geodesic generators of $\widetilde{\mathcal{H}}_1$, and similarly, $\Psi_3 = r \cdot \overline{\eth}\Psi_2$ along the null geodesic generators of $\widetilde{\mathcal{H}}_2$.

The next obvious question is whether we do really have a blow up also in terms of measurable quantities or what is indicated above is simply the consequence of an inappropriate choice of a frame field along the null geodesic generators of the bifurcate event horizon. In virtue of the following lemma, along with the discussion following it, the indicated possible blowing up of the Weyl spinor components, e.g., that of Ψ_0 and Ψ_1 along the generators of $\mathcal{H}_1$, are always related to true "*parallelly propagated*", *p.p.*, curvature singularities.

Lemma 4.4 *l^a is parallelly propagated with respect to n^a on $\widetilde{\mathcal{H}}_1$.*

Proof Notice first that $n^e\nabla_e l^a = 0$ if the all the contractions $n_a n^e\nabla_e l^a$, $l_a n^e\nabla_e l^a$ and $m_a n^e\nabla_e l^a$ vanish. The first vanishes since $n_a n^e\nabla_e l^a = -l_a n^e\nabla_e n^a$ and n^a is parallelly propagated with respect to itself on $\widetilde{\mathcal{H}}_1$. The second contraction is zero because l^a is null everywhere. Finally, the vanishing of the third contraction on $\widetilde{\mathcal{H}}_1$ is guaranteed by Lemma 3.2. □

Since along the generators of $\widetilde{\mathcal{H}}_1$ any parallelly propagated spacelike unit vector field—which is orthogonal to l^a and n^a—can be given by a linear combination of the "unite" norm spacelike vectors $x^a = \frac{1}{\sqrt{2}}(m^a + \overline{m}^a)$ and $y^a = \frac{1}{\sqrt{2}}i(m^a - \overline{m}^a)$ with bounded coefficients, everywhere along the generators of $\widetilde{\mathcal{H}}_1$, we have that the above indicated blowing up of the Weyl spinor components Ψ_0, Ψ_1 and Ψ_3, Ψ_4 are always associated with true "parallelly propagated" curvature singularities, which possibility was already pointed out and discussed in [23] (see Remark 6.2 for more details).

Notice that the existence of the indicated *p.p.* curvature singularity is not associated with the incompleteness of the null geodesic generators of the bifurcate horizon. It is also interesting that there is a difference between the possible strengths of the associated curvature blows up. While Ψ_1 and Ψ_3 might blow up only linearly when a blow up occur in case of Ψ_0 and Ψ_4 that has to be quadratic with respect to the associated synchronised affine parameters.

In virtue of the above discussion, we could also say that the desired global Gaussian null coordinates can exists whenever the associated stationary black hole spacetime admits the existence only of the mild type of *p.p.* curvature singularities at the future and past ends of the bifurcate event horizon. This, however, in particular implies that even the Kerr solution admits the associated mild type of *p.p.* curvature blow up, the existence of which is not at all incompatible with the vanishing of the Weyl spinor components Ψ_0' and Ψ_1' with respect to the null tetrad, $\{l'^a, n'^a, m'^a, \overline{m}'^a\}$, yielded by the null rotation (4.3) of $\{l^a, n^a, m^a, \overline{m}^a\}$, which we applied in the proof of Lemma 4.2. To see this, notice that, in the generic case, n'^a is neither tangent to the generators of $\widetilde{\mathcal{H}}_1$, nor it is parallel with respect to n^a.

Finally, we would like to emphasise that the above discussed *p.p.* curvature singularities are not strong enough to be "*scalar curvature singularity*", i.e., neither of the scalar invariants of the Weyl tensor blows up along the generators of the event horizon. The existence of the above described *p.p.* curvature singularities could also be interpreted as having a sort of warning signal indicating that certain *tidal-force* effects increase "in time", along the black hole event horizon. Thereby, if, for certain reasons, one needs to enter the black hole region, e.g., of the Kerr black hole probably the sooner is the better.

5 The electrovac black hole spacetimes

This section is to discuss the differences which show up in case of the electrovac black hole spacetimes with non-zero cosmological constant. We start by providing the explicit form of the "reduced field equations", which similarly to the vacuum case, consist of some of the Newman-Penrose and Maxwell equations or suitable linear combinations of pairs of the Newman-Penrose and Maxwell equations [17].

Recall, first, that electromagnetic field, F_{ab}, can very effectively be represented in the Newman-Penrose formalism [17] by making use of the contractions

$$\phi_0 = F_{ab} l^a m^b \tag{5.1}$$

$$\phi_1 = \frac{1}{2} F_{ab}\left(l^a n^b + \overline{m}^a m^b\right) \tag{5.2}$$

$$\phi_2 = F_{ab}\overline{m}^a n^b\,, \tag{5.3}$$

while the energy momentum tensor, as given by (2.3), can be represented via the Ricci spinor components, Φ_{ij}, where the indices i,j take the values $0,1,2$—for their generic definitions see (NP.4.3b)—which for the considered electrovac case, (2.3), can be given as

$$\Phi_{ij} = 2\,\phi_i\overline{\phi}_j\,. \tag{5.4}$$

By making use of these variables the reduced Einstein's equations, relevant for the electrovac case with cosmological constant, $\widetilde{\Lambda} = 6\Lambda$, read as

$$\mathrm{D}\,\xi^A = \rho\,\xi^A + \sigma\,\bar{\xi}^A \tag{EM.1}$$
$$\mathrm{D}\,\omega = \rho\,\omega + \sigma\,\overline{\omega} - \tau \tag{EM.2}$$
$$\mathrm{D}X^A = \tau\,\bar{\xi}^A + \bar{\tau}\,\xi^A \tag{EM.3}$$
$$\mathrm{D}\,U = \tau\,\overline{\omega} + \overline{\tau}\,\omega - (\gamma+\overline{\gamma}) \tag{EM.4}$$
$$\mathrm{D}\,\rho = \rho^2 + \sigma\,\overline{\sigma} + \Phi_{00} \tag{EM.5}$$
$$\mathrm{D}\,\sigma = 2\rho\,\sigma + \Psi_0 \tag{EM.6}$$
$$\mathrm{D}\,\tau = \tau\,\rho + \overline{\tau}\,\sigma + \Psi_1 + \Phi_{01} \tag{EM.7}$$
$$\mathrm{D}\,\alpha = \rho\,\alpha + \beta\,\overline{\sigma} + \Phi_{10} \tag{EM.8}$$
$$\mathrm{D}\,\beta = \alpha\,\sigma + \rho\,\beta + \Psi_1 \tag{EM.9}$$
$$\mathrm{D}\,\gamma = \tau\,\alpha + \overline{\tau}\,\beta + \Psi_2 - \Lambda + \Phi_{11} \tag{EM.10}$$
$$\mathrm{D}\,\lambda = \rho\,\lambda + \overline{\sigma}\,\mu + \Phi_{20} \tag{EM.11}$$
$$\mathrm{D}\,\mu = \rho\,\mu + \sigma\,\lambda + \Psi_2 + 2\,\Lambda \tag{EM.12}$$
$$\mathrm{D}\,\nu = \overline{\tau}\,\mu + \tau\,\lambda + \Psi_3 + \Phi_{21} \tag{EM.13}$$
$$\begin{aligned}\Delta\,\Psi_0 - \delta\,(\Psi_1{+}\Phi_{01}) + \mathrm{D}\,\Phi_{02} &= (4\,\gamma - \mu)\,\Psi_0 - 2\,(2\,\tau + \beta)\,\Psi_1 + 3\,\sigma\,\Psi_2 \\ &\quad -\overline{\lambda}\,\Phi_{00} - 2\,\beta\,\Phi_{01} + 2\,\sigma\,\Phi_{11} + \rho\,\Phi_{02}\end{aligned} \tag{EM.14}$$
$$\begin{aligned}&\Delta\,(\Psi_1 - \Phi_{01}) + \mathrm{D}\,(\Psi_1{-}\Phi_{01}) - \delta\,(\Psi_2 + 2\,\Lambda) + \delta\,\Phi_{00} - \overline{\delta}\,\Psi_0 + \overline{\delta}\,\Phi_{02} = \\ &\quad +(\nu - 4\,\alpha)\,\Psi_0 + 2\,(\gamma + 2\,\rho - \mu)\,\Psi_1 - 3\,\tau\,\Psi_2 + 2\,\sigma\,\Psi_3 \\ &\quad +(2\,\tau - \overline{\nu})\,\Phi_{00} + 2\,(\overline{\mu} - \gamma - \rho)\,\Phi_{01} - 2\,\sigma\,\Phi_{10} + 2\,\tau\,\Phi_{11} + (3\,\alpha - \overline{\beta})\,\Phi_{02} - 2\,\rho\,\Phi_{12}\end{aligned} \tag{EM.15}$$
$$\begin{aligned}&\Delta\,(\Psi_2 + 2\,\Lambda) + \mathrm{D}\,(\Psi_2 + 2\,\Lambda) - \delta\,(\Psi_3 + \Phi_{21}) - \overline{\delta}\,(\Psi_1 + \Phi_{01}) + \Delta\,\Phi_{00} + \mathrm{D}\,\Phi_{22} = \\ &\quad -\lambda\,\Psi_0 + 2\,(\nu - \alpha)\,\Psi_1 + 3\,(\rho - \mu)\,\Psi_2 - 2\,\overline{\alpha}\,\Psi_3 + \sigma\,\Psi_4 \\ &\quad +(2\,\gamma + 2\,\overline{\gamma} - \overline{\mu})\,\Phi_{00} - 2\,(\alpha{+}\overline{\tau})\,\Phi_{01} - 2\,\tau\,\Phi_{10} + 2\,(\rho - \mu)\,\Phi_{11} \\ &\quad -\overline{\lambda}\,\Phi_{20} + \overline{\sigma}\,\Phi_{02} + 2\,\beta\,\Phi_{21} + \rho\,\Phi_{22}\end{aligned} \tag{EM.16}$$
$$\begin{aligned}&\Delta\,(\Psi_3{-}\Phi_{21}) + \mathrm{D}\,(\Psi_3{-}\Phi_{21}) - \delta\,\Psi_4 - \overline{\delta}\,(\Psi_2 + 2\,\Lambda) + \delta\,\Phi_{20} + \overline{\delta}\,\Phi_{22} = \\ &\quad -2\,\lambda\,\Psi_1 + 3\,\nu\,\Psi_2 - 2\,(\gamma + 2\,\mu - \rho)\,\Psi_3 + (4\,\beta - \tau)\,\Psi_4 \\ &\quad +2\,\mu\,\Phi_{10} - (2\,\beta - 2\,\overline{\alpha} + \nu)\,\Phi_{20} - 2\,\nu\,\Phi_{11} + 2\,\lambda\,\Phi_{12} + 2\,(\gamma + \overline{\mu} - \rho)\,\Phi_{21} - \overline{\tau}\,\Phi_{22}\end{aligned} \tag{EM.17}$$
$$\begin{aligned}\mathrm{D}\,\Psi_4 - \overline{\delta}\,(\Psi_3 + \Phi_{21}) + \Delta\,\Phi_{20} &= -3\,\lambda\,\Psi_2 + 2\,\alpha\,\Psi_3 + \rho\,\Psi_4 \\ &\quad +2\,\nu\,\Phi_{10} - 2\,\lambda\,\Phi_{11} - (2\,\gamma - 2\,\overline{\gamma} + \overline{\mu})\,\Phi_{20} - 2\,(\overline{\tau} - \alpha)\,\Phi_{21} + \overline{\sigma}\,\Phi_{22}\end{aligned} \tag{EM.18}$$

These equations have to be augmented by the "reduced Maxwell" equations which read as

$$\Delta\phi_0 - \delta\phi_1 = (2\,\gamma - \mu)\,\phi_0 - 2\,\tau\,\phi_1 + \sigma\,\phi_2 \tag{EM.19}$$
$$\Delta\phi_1 + \mathrm{D}\phi_1 - \delta\phi_2 - \overline{\delta}\phi_0 = (\nu - 2\,\alpha)\,\phi_0 + 2\,(\rho - \mu)\,\phi_1 - (\tau - 2\,\beta)\,\phi_2 \tag{EM.20}$$
$$\mathrm{D}\phi_2 - \overline{\delta}\phi_1 = -\lambda\,\phi_0 + \rho\,\phi_2\,. \tag{EM.21}$$

It can be proved, completely parallel to the argument outlined in Subsection 3.4 for the vacuum case, that (EM1) - (EM21) comprise a determined system for the "21-dimensional" vector valued variable

$$\mathbb{V}_{EM} = (\xi^A, \omega, X^A, U; \rho, \sigma, \tau, \alpha, \beta, \gamma, \lambda, \mu, \nu; \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4; \phi_0, \phi_1, \phi_2)\,. \tag{5.5}$$

These equations can also be shown to be "as good as" the full set of the Newman-Penrose and Maxwell equations. Moreover, when written out these equations in Gaussian null coordinates (u, r, x^3, x^4) they possess the form of a first order quasilinear symmetric hyperbolic system, i.e., it can be justified that they read as

$$(\mathbb{A}_{EM})^\mu \cdot \partial_\mu \mathbb{V}_{EM} + \mathbb{B}_{EM} = 0, \tag{5.6}$$

where the matrices $(\mathbb{A}_{EM})^\mu$ and $\mathbb{B}_{EM}$ smoothly depend on $\mathbb{V}_{EM}$ and $\overline{\mathbb{V}}_{EM}$, moreover, the matrices $(\mathbb{A}_{EM})^\mu$ are Hermitian and the combination $(\mathbb{A}_{EM})^\mu(n_\mu + l_\mu)$ is positive definite.

In addition, it can also be shown that the reduced set of initial data includes, beside the usual data relevant for the vacuum configurations, the value of the cosmological constant, $\widetilde{\Lambda} = 6\Lambda$, and the specification of the Maxwell spinor components ϕ_2 on $\widetilde{\mathcal{H}}_1$, ϕ_0 on $\widetilde{\mathcal{H}}_2$ and ϕ_1 at the bifurcation surface, $\mathcal{Z}$, i.e. it is given as

$$(\mathbb{V}_{EM})_0^{red} = \{\rho, \sigma, \mu, \lambda, \tau\,;\, \xi^A; \phi_1\}|_{\widetilde{\mathcal{Z}}} \cup \{\Psi_4; \phi_2\}|_{\widetilde{\mathcal{H}}_1} \cup \{\Psi_0; \phi_0\}|_{\widetilde{\mathcal{H}}_2} \cup \{\Lambda \in \mathbb{R}\}\,. \tag{5.7}$$

It is straightforward to justify that, just like in the vacuum case, our domain of dependence argument given in Subsection 3.5 does also apply to the selected electrovac case. Summarising then all the above claims the following statement can be shown to be true.

Theorem 5.1 *In the characteristic initial value problem to any 'reduced initial data set', $(\mathbb{V}_{EM})_0^{red}$, on $\mathcal{H}_1 \cup \mathcal{H}_2$, there always exists a unique solution, $\mathbb{V}_{EM}$, everywhere in the associated elementary spacetime region $\mathcal{O}$, to the electrovac Einstein-Maxwell equations.*

The determination of a full initial data set $(\mathbb{V}_{EM})_0$ on $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$ can be done completely parallel to the construction applied in Subsection 3.7. The relevant results are collected in Table 2 below.

In the electrovac case most of the results concerning the existence of shear free null geodesic congruences, which have been derived previously only for the vacuum case, remain valid with some slight modifications due to the presence of the electromagnetic field and the cosmological constant. For instance, the conclusion of Lemma 4.3 implying that

$$\Psi_2 = \eta' + \chi' \cos(\theta), \tag{5.8}$$

with some $\eta', \chi' \in \mathbb{R}$, remains intact. Notice, however, that in the present case the Gaussian curvature, $\mathcal{K}_{\mathcal{G}}$, of the 2-dimensional spacelike surface $\mathcal{Z}$ takes the form

$$\mathcal{K}_{\mathcal{G}} = \left\{-2\,\Psi_2 + 4\,\phi_1\overline{\phi}_1 + 2\,\Lambda\right\}\big|_{\mathcal{Z}}\,. \tag{5.9}$$

This, for instance, when we would like to exclude the possibility of a *p.p.* blowing up of the electromagnetic field, or more specifically, that of ϕ_0 along the generators of $\widetilde{\mathcal{H}}_1$, as well as, that of ϕ_2 along the generators of $\widetilde{\mathcal{H}}_2$—this could happen only whenever $\eth\phi_1 \neq 0$—only the value of the constant η is modified in (4.8).

The most significant change which shows up in the related analysis is due to the presence of a non-zero cosmological constant, $\widetilde{\Lambda} = 6\Lambda$. More specifically, to guarantee that the expansion of the "outgoing" null congruences at $\mathcal{H}_1$ be non-negative we need to adjust the value of the cosmological constant $\widetilde{\Lambda}$ so that the inequality $\Psi_2 + 2\Lambda \leq 0$ should hold throughout $\mathcal{Z}$. This implies, in particular, that $\widetilde{\Lambda} \leq -3\,\Psi_2$ has to hold there. Since, in general, Ψ_2 seems to be negative, at least for physically reasonable black hole configurations possessing non-negative mass, the latter inequality seem to impose only a positive upper bound on the possible value of the cosmological constant $\widetilde{\Lambda}$.

$\widetilde{\mathcal{H}}_1$	$\widetilde{\widetilde{\mathcal{Z}}}$	$\widetilde{\mathcal{H}}_2$
$\rho = -u \cdot (\Psi_2 + 2\Lambda)$	$\rho = 0$	$\rho = 0$
$\mu = 0$	$\mu = 0$	$\mu = r \cdot (\Psi_2 + 2\Lambda)$
$\sigma = \lambda = \tau = 0$	$\sigma = \lambda = \tau = 0$	$\sigma = \lambda = \tau = 0$
$\Delta\alpha = \Delta\beta = 0$	$\alpha, \beta:\ \tau = \overline{\alpha} + \beta = 0$	$\mathrm{D}\alpha = \mathrm{D}\beta = 0$
$\Delta\Psi_2 = 0$	$\xi^A, \phi_1, \Lambda \ \&\ \alpha, \beta \to \Psi_2$	$\mathrm{D}\Psi_2 = 0$
$\phi_0 = u \cdot \eth\phi_1$	$\phi_0 = 0$	$\phi_0 = 0$
$\Delta\phi_1 = 0$	ϕ_1	$\mathrm{D}\phi_1 = 0$
$\phi_2 = 0$	$\phi_2 = 0$	$\phi_2 = r \cdot \overline{\eth}\phi_1$
$\Psi_0 = \frac{1}{2}u^2\,\eth^2\Psi_2$	$\Psi_0 = 0$	$\Psi_0 = 0$
$\Psi_1 = u \cdot \eth\Psi_2$	$\Psi_1 = 0$	$\Psi_1 = 0$
$\Psi_3 = 0$	$\Psi_3 = 0$	$\Psi_3 = r \cdot \overline{\eth}\Psi_2$
$\Psi_4 = 0$	$\Psi_4 = 0$	$\Psi_4 = \frac{1}{2}r^2\,\overline{\eth}^2\Psi_2$
(gauge) $\nu = 0 \quad \to$	$\nu = 0 \quad \to$	$\nu = \frac{1}{2}r^2 \cdot \left(\overline{\eth}\Psi_2 + 2\,\overline{\eth}\phi_1 \cdot \overline{\phi_1}\right)$
(gauge) $\gamma = 0 \quad \to$	$\gamma = 0 \quad \to$	$\gamma = r \cdot (\Psi_2 - \Lambda + \Phi_{11})$

Table 2: The full initial data set $(\mathbb{V}_{EM})_0$ relevant for a stationary electromagnetic black hole spacetime, on the intersecting null hypersurfaces $\widetilde{\mathcal{H}}_1 \cup \widetilde{\mathcal{H}}_2$.

6 Final remarks

In the traditional investigations of stationary black hole spacetimes it is always assumed that they possess some sort of asymptotic structure hence the corresponding spacetimes are assumed to be asymptotically flat or asymptotically (locally) anti-de-Sitter. As it was demonstrated above we may drop all the requirements concerning the asymptotic structure and carry out the "semi-quasi-local" investigations of stationary black hole spacetimes in the corresponding geometrical setting. This strategy, in principle, could enlarge the associated configuration space—which will be referred as $\mathcal{Q}_{st.b.h.}$—considerably since it may happen that there are stationary (electrovac) black holes which satisfy all the technical assumptions imposed in Subsection 2 and, on the other hand, the associated domain of outer communications are not asymptotically flat or asymptotically (locally) anti-de-Sitter.

As we have seen in our investigations the members of the associated enlarged configuration space, $\mathcal{Q}_{st.b.h.}$, can be characterised—once an appropriate gauge choice has been made—by a complex valued vector field ξ^A on the bifurcation surface which is in direct correspondence with the geometry of the null geodesic generators of the event horizon. It is quite conspicuous that all the new members of the enlarged configuration space do nicely fit to the framework of spacetimes possessing "*a non-expanding-*"', "*a (weakly) isolated-*" or "*a weakly isolated rigidly rotating-*" horizon, the notions of which were introduced (and evolved) by Ashtekar and his co-workers. However, we need to be very careful because all those spacetimes which belong to the enlarged configuration space $\mathcal{Q}_{st.b.h.}$ are still very much of the same fashion as the customary stationary black hole solutions to the electrovac Einstein's equations since they all possess a "horizon Killing vector field" on the entire of the associated elementary spacetime region. Since surface gravity, $\kappa_\circ$, was assumed to be positive we know that the horizon Killing vector field is timelike on the domain of outer communication side, while it is spacelike on the black hole region side, at least, in a sufficiently small open neighbourhood of the event horizon. Consequently, however tempting it was to consider the generic members of $\mathcal{Q}_{st.b.h.}$ as being spacetimes with an isolated horizon, it is more appropriate to consider them as being stationary black hole spacetimes.

Concerning these generic stationary black hole solutions in $\mathcal{Q}_{st.b.h.}$—according to our current understanding—either of the following two complementary cases can occur. First, if they can be shown to smoothly extend beyond the boundary of the elementary spacetime region—which is also the boundary where the loss of the horizon Killing vector field and also that of analyticity happens—they will be welcome as the the missing "rumpled hairy" black holes. Notice that here a fibre of hair was tacitly represented by a past directed (it could equally be chosen to be future directed) null geodesic while the entire hair of a black hole is represented by the 3-parameter family of null geodesic congruences starting at the points of the event horizon with past directed null tangent vector field $\mathfrak{L}^a$, as it was described above. According to this picture even the customary asymptotically flat or asymptotically (locally) anti-de-Sitter stationary electrovac black holes—which have been considered for long to have "no hair"—get to be hairy, although, their hair is so much well-set that it is easy to pass by without perceiving their existence. In the second complementary case, i.e., whenever the generic spacetimes in $\mathcal{Q}_{st.b.h.}$ do not extend smoothly beyond the boundary of their elementary spacetime regions, the situation is more severe since then there might be true physical singularities associated with these spacetimes. If this latter prospect would occur—unfortunately, we have not been able to exclude this possibility in either way—then, as the associated singularities are located on the "domain of outer communication side", their existence would shed light on some interesting issues related to the validity of the "*cosmic censor hypotheses*" of Penrose. Actually, according to our current understanding, the associated situation would be even more exciting yielding a shaking of all the cornerstones of classical general relativity. Therefore, it is, in fact, the most important open issue whether the *complete* lack of the global Gaussian null coordinates are related to true physical singularities or not.

Until the above issue has not been settled let us to be pragmatic, or, what is closer to our wish, to be more "positivist" and assume that the generic black holes spacetimes in $\mathcal{Q}_{st.b.h.}$ will be regularly extendible beyond the boundary of their maximal elementary spacetime region, even though, the horizon Killing vector field will not extend beyond this boundary. Then the mathematical framework, see Subsections 3.4 and 3.5, we applied in this paper in carrying out all the investigations—namely, the characteristic initial data value formulation of general relativity as it was developed by Helmut Friedrich, based on the application of the Newman-Penrose formalism—could be suitable to carry out investigations of both linear and non-linear perturbations of these spacetimes. The corresponding perturbative approach could be extremely powerful, in the particular cases, when the customary asymptotically flat or asymptotically (locally) anti-de-Sitter stationary black hole spacetimes—which possess globally well-behaving Gaussian null coordinates—were chosen to be the seed solutions because then, as opposed to all the former perturbative approaches, the associated investigations could be carried out on the entire of these black hole spacetimes—including the black hole and domain of outer communication regions—simultaneously. Notice that within the proposed framework, beside being able to investigate several problems with immediate relevance for gravitational wave physics, hopefully some hints about the aforementioned more fundamental conceptual issue could also be acquire.

Recall that as an interesting "byproduct" of our main argument apparently we could reproduce the entire content of the black hole uniqueness argument. In fact, what could be done seems to be really powerful since in the "orthodox" approach attention is restricted to the class of asymptotically flat or asymptotically (locally) anti-de-Sitter stationary electrovac black hole spacetimes (see, e.g., Refs. [3, 13, 14, 4, 2]), while in the new framework the entire of $\mathcal{Q}_{st.b.h.}$ could be scanned in identifying the "unique" solutions. This unexpected success of the new approach is very suggestive. It indicates that we could, and in fact, we probably should reorganise the mathematical framework related to the black hole uniqueness argument. It seems to be really appropriate to do this now before submerging too deep into the much larger configuration space of stationary black hole solutions relevant for higher dimensional generalisations of Einstein's theory of gravity.

It is also worth keeping in mind that the configuration space $\mathcal{Q}_{st.b.h.}$ is pretty large. Contrary to our former understanding, the asymptotically flat or asymptotically (locally) anti-de-Sitter stationary electrovac black hole spacetimes seem to comprise only a subset of zero measure within $\mathcal{Q}_{st.b.h.}$. Since all the physical degrees of freedom can be given in terms of the induced 2-metric on the bifurcation surface $\mathscr{Z}$ it is tempting to think that we could also get closer to the missing part of classical black hole thermodynamics, more specifically, to the microcannonical determination of the black hole entropy. In this respect, it would be interesting to know whether by representing ξ^A, on the bifurcation surface $\mathscr{Z}$, say in terms of *spin weighted spherical harmonics* there could be any hope in determining a microcannonical entropy in fashion similar to those proposed in Refs. [1, 15, 29]. The fundamental results of [27] indicate that this might be possible to be done even in case of pure vacuum general relativity based on the quasi-local canonical framework proposed in that reference.

In virtue of the above discussions it seems also to be quite appropriate to think of the bifurcation surface as the unique compact "carrier" of the preimage of the entire associated elementary spacetime region, which can be built up by making use of the field equations once the carrier is provided. In this respect it seems to be completely suitable to think of the bifurcation surface as a "portable" holograph. While, in the context of the investigated special black hole spacetimes the associated holograph can be considered, in some extent to be global—it is certainly global in case of asymptotically flat or asymptotically (locally) anti-de-Sitter stationary black hole spacetimes—apparently a more generic notion of (local) holographic principle can also be associated with any spacetime in Einstein's theory of gravity. The existence and properties of the corresponding holographic principle will be discussed in our coming paper.

Let us finally mention that all the above reported findings of our investigations—forwarded occasionally in terms of very technical terms and statements—could also be expressed in the following "simple-minded" way. By moving along a (sufficiently smooth) imaginary curve in $\mathcal{Q}_{st.b.h.}$—the points of which can be represented by the associated spherical bifurcation surfaces, or equivalently, by the carriers of the holographs—we can listen to the "music of spheres" mentioned first in scientific context by Pythagoras and his followers.

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